Announcements:

Midterm 1 graded

Median 63/80

Mean: 56.4/80

Std. dev: 21.62

Q1: 79%

Q2: 68%

Q3: 58%

Q4: 77%

Gradelines:

A-/A: 68 to 80

B+/B/B-: 50 to $68-\varepsilon$

C+/C/C-: 30 to $50-\varepsilon$

D+/D/D-: 13 to $30-\varepsilon$

Sol'ns posted to website

"Where do I stand" spreadsheet posted to website

disclaimers!

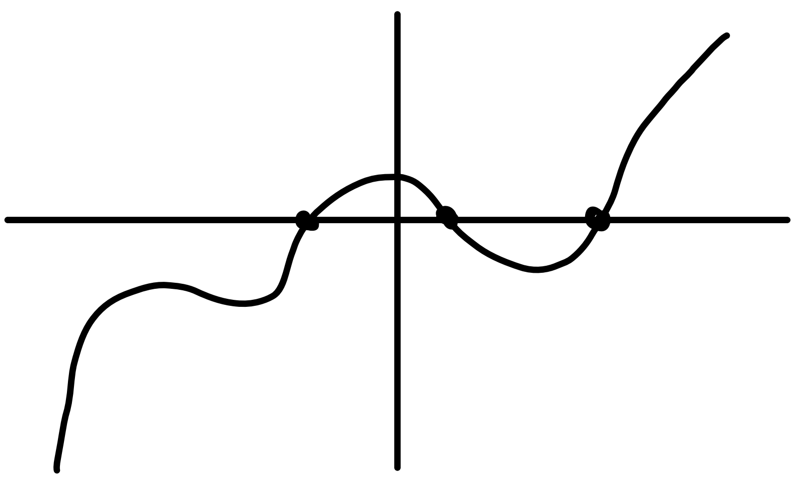
Fundamental Thm. of Algebra (Gauss): $\mathbb{C}$ is alg. closed

Cor: If $F \subseteq \mathbb{C}$, then $\overline{F} \subseteq \mathbb{C}$, so e.g. $\overline{\mathbb{Q}}$ = set of alg. numbers

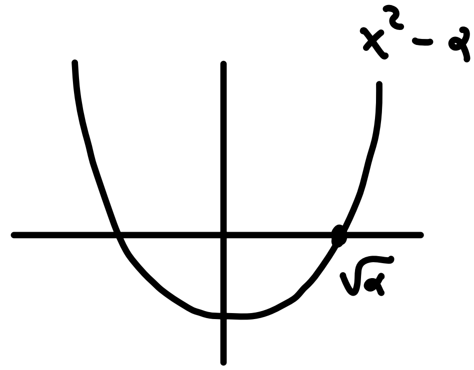
Pf sketch using Galois theory:

Two analytic consequences of the Intermediate Value Theorem

(A) Every odd degree poly. in $\mathbb{R}[x]$ has a root in $\mathbb{R}$



(B) Every $\alpha \in \mathbb{R}_{\geq 0}$ has a sqrt. $\sqrt{\alpha} \in \mathbb{R}_{\geq 0}$



Let $f(x) \in \mathbb{R}[x]$, f irred., $n := \deg f$.

WTS: f has a root in $\mathbb{C}$.

Let $K := Sp_{\mathbb{R}} f$

$$\begin{tikzcd}[column sep=small]
 & K(i) \arrow[dl, dash] \arrow[dr, dash, "1 \text{ or } 2"] & \\
\mathbb{C} \arrow[dr, dash, "2"'] & & K \arrow[dl, dash, "n"] \\
 & \mathbb{R} &
\end{tikzcd}$$

Galois theory gives us detailed information about intermediate fields.

In this case,

$$\begin{tikzcd}
K(i) \arrow[d, dash, "\text{power of } 2"] \\
\exists\, E \arrow[d, dash, "\text{odd}"] \\
\mathbb{R}
\end{tikzcd}$$

impossible by (A) unless $E = \mathbb{R}$

So we have

$$\begin{tikzcd}[row sep=small]
 & & K(i) \arrow[d, dash, "2"] \\
 & \exists \arrow[r] & E_k \arrow[d, dash, "2"] \\
K(i) \arrow[dd, dash, "\text{power of } 2"'] & & \vdots \arrow[d, dash, "2"] \\
 & \arrow[r, squiggly, "\text{more Galois theory}"] & E_1 \arrow[d, dash, "2"] \\
\mathbb{R} & & \mathbb{C} \arrow[d, dash, "2"] \\
 & & \mathbb{R}
\end{tikzcd}$$

impossible by (B) unless $\mathbb{C} = K(i)$

$\square$

This suggests that we need to know more detailed information about field ext'ns

Separable extensions

Let $f(x) \in F[x]$; over $K = Sp_F f$, we have

$$f(x) = (x-\alpha_1)^{n_1} \cdots (x-\alpha_k)^{n_k}$$

distinct

n_i: multiplicity of α_i

α_i is simple if $n_i = 1$

α_i is multiple if $n_i > 1$

Def: f is separable if all its roots$/K$ are simple. Otherwise it's inseparable.

Ex: $x^2 - 2 = (x+\sqrt{2})(x-\sqrt{2})$

$x^n - p = (x - \sqrt[n]{p})(x - \zeta_n \sqrt[n]{p}) \cdots (x - \zeta_n^{n-1} \sqrt[n]{p})$

prime

$x^2 + 1 = (x+i)(x-i)$

$x^2 - 1 = (x+1)(x-1)$

all separable

Non-ex:

a) $x^2 + 2x + 1 = (x+1)^2 \in \mathbb{Q}[x]$

-1 is a multiple root

b) $f(x) = x^2 + t \in \mathbb{F}_2(t)[x]$

irred. by Eisenstein, using the prime $t \in \underbrace{\mathbb{F}_2[t]}_{\text{UFD}}$

or rat'l root thm. for similar reasons

Let $K = Sp\, f$, and let $\alpha \in K$ be a root of x^2+t i.e. $\alpha^2 = t$

$$(x-\alpha)^2 = x^2 - 2\alpha x + t = x^2 + t$$

So f is <u>not</u> separable

Thm: If

a) char $F = 0$ or

b) F is finite,

then every irred. $f(x) \in F[x]$ is separable.

Def: The derivative of $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in F[x]$ is

$$Df(x) = n a_n x^{n-1} + \cdots + 2a_2 x + a_1 \in F[x]$$

No calculus needed! Product/chain rules hold as usual

Separability Criterion: Let $f(x) \in F[x]$.

a) α is a multiple root of f $\iff$ α is a root of f and Df

b) $f(x)$ is separable $\iff$ $\gcd(f, Df) = 1$

Pf: a) $\Rightarrow$) $f(x) = (x-\alpha)^n g(x)$ $\quad n \geq 2$

$$Df = n(x-\alpha)^{n-1} g(x) + (x-\alpha)^n Dg$$
$$= (x-\alpha)\left[n(x-\alpha)^{n-2} g(x) + (x-\alpha)^{n-1} Dg\right] \Rightarrow Df(\alpha) = 0$$

$\Leftarrow$) $f(x) = (x-\alpha) h(x)$

$$Df = h(x) + (x-\alpha) Dh(x)$$

$$0 = Df(\alpha) = h(\alpha) + (\alpha-\alpha) Dh(\alpha) \Rightarrow h(\alpha) = 0 \Rightarrow (x-\alpha)^2 \mid f.$$

b) Will show for $p, q \in F[x]$ that

$\gcd(p,q) = 1 \iff p, q$ have no common roots in an ext'n field K where they split completely

Case p, q have common root α: then p, q are both divisible by $m_{\alpha,F}(x)$

Case no common root: If $\gcd(p,q) = r(x) \in F[x]$ nonconst. then any root of $r(x)$ in K is a common root of p & q. $\square$