Last time: Every poly. $f(x) \in F[x]$ has a <u>splitting field</u> $K := Sp_F f$, which is unique up to isomorphism.

f factors into linear factors over K, but over no smaller field ext'n

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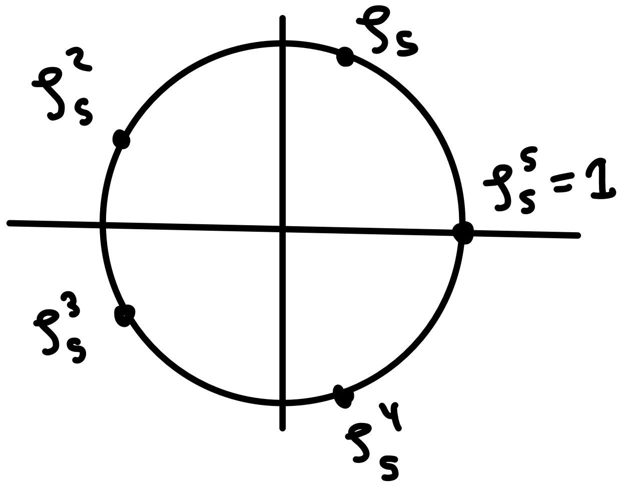
Def/Ex: Let ζ_n (usually $\zeta_n = e^{2\pi i/n}$) be a primitive nth root of 1.

The field $\mathbb{Q}(\zeta_n)$ is the <u>cyclotomic field</u> of nth roots of 1



$$x^n - 1 = (x-1)(x-\zeta_n)(x-\zeta_n^2)\cdots(x-\zeta_n^{n-1})$$

$$= (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

and $1, \zeta_n, \ldots, \zeta_n^{n-1} \in \mathbb{Q}(\zeta_n)$

So $\mathbb{Q}(\zeta_n)$ is the splitting field for $x^n - 1$

$[\mathbb{Q}(\zeta_n) : \mathbb{Q}] \leq n-1$ w/ equality iff p: prime (HW3 #4)

Ex: $f(x) = x^p - 2 \in \underbrace{\mathbb{Q}[x]}_{\text{default}}$, p : prime

$$f(x) = (x - \sqrt[p]{2})(x - \zeta_p \sqrt[p]{2}) \cdots (x - \zeta_p^{p-1} \sqrt[p]{2})$$

unique pos. real
pth root of 2

Splitting field: $S_p(x^p - 2) = \mathbb{Q}(\sqrt[p]{2}, \zeta_p)$

Composite ext'n:

$$[\mathbb{Q}(\sqrt[p]{2}, \zeta_p) : \mathbb{Q}] \leq [\mathbb{Q}(\sqrt[p]{2}) : \mathbb{Q}][\mathbb{Q}(\zeta_p) : \mathbb{Q}]$$

Tower Law:

$$p = [\mathbb{Q}(\sqrt[p]{2}) : \mathbb{Q}] \mid [\mathbb{Q}(\sqrt[p]{2}, \zeta_p) : \mathbb{Q}]$$

$$p-1 = [\mathbb{Q}(\zeta_p) : \mathbb{Q}] \mid [\mathbb{Q}(\sqrt[p]{2}, \zeta_p) : \mathbb{Q}]$$

coprime

So

$$[\mathbb{Q}(\sqrt[p]{2}, \zeta_p) : \mathbb{Q}] = p(p-1)$$

Ok, we can get one poly. to split. What about <u>all</u> polys.?

Def: We'll use this notation

a) $\overline{F}$ is an algebraic closure of F if $\overline{F}/F$ is alg. and every $f(x) \in F[x]$ splits completely in $\overline{F}[x]$

(equivalently, every nonconstant $f(x) \in F[x]$ has a root in $\overline{F}$)

b) K is alg. closed if $\overline{K} = K$

Prop: Alg. closure $\Longrightarrow$ alg. closed
(i.e. If $K = \overline{F}$, $K = \overline{K}$)

Pf:

$$\underbrace{F \underset{\text{alg.}}{\subseteq} K = \overline{F} \underset{\text{alg.}}{\subseteq} \overline{K}}_{\text{alg.}}$$

So every elt. of $\overline{K}$ is a root of some poly $/F$.

$\square$

Thm: Every field F has an alg. closure $\overline{F}$, which is unique up to isom.

Pf: see D&F Props. 30 & 31

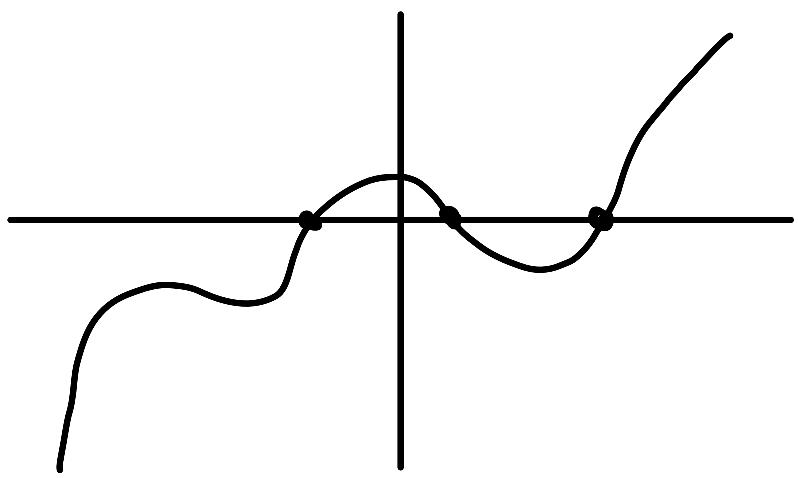
Fundamental Thm. of Algebra (Gauss): $\mathbb{C}$ is alg. closed

Cor: If $F \subseteq \mathbb{C}$, then $\overline{F} \subseteq \mathbb{C}$, so e.g. $\overline{\mathbb{Q}}$ = set of alg. numbers

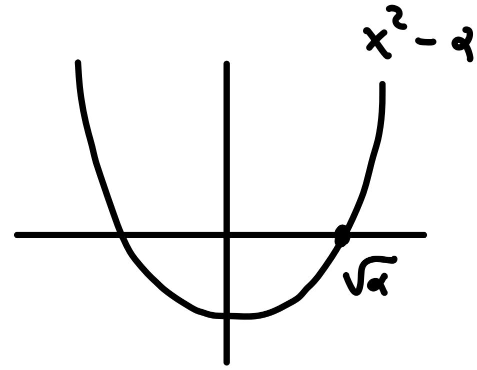
Pf sketch using Galois theory:

Two analytic consequences of the Intermediate Value Theorem

(A) Every odd degree poly. in $\mathbb{R}[x]$ has a root in $\mathbb{R}$



(B) Every $\alpha \in \mathbb{R}_{\geq 0}$ has a sqrt. $\sqrt{\alpha} \in \mathbb{R}_{\geq 0}$



Let $f(x) \in \mathbb{R}[x]$, f irred., $n := \deg f$.

WTS: f has a root in $\mathbb{C}$.

Let $K := Sp_{\mathbb{R}} f$

$$\begin{tikzcd}[column sep=small]
 & K(i) \arrow[dl, dash] \arrow[dr, dash, "1 \text{ or } 2"] & \\
\mathbb{C} \arrow[dr, dash, "2"'] & & K \arrow[dl, dash, "n"] \\
 & \mathbb{R} &
\end{tikzcd}$$

Galois theory gives us detailed information about intermediate fields.

In this case,

$$\begin{tikzcd}
K(i) \arrow[d, dash, "\text{power of } 2"] \\
\exists\, E \arrow[d, dash, "\text{odd}"] \\
\mathbb{R}
\end{tikzcd}$$

impossible by (A) unless $E = \mathbb{R}$

So we have

$$\begin{tikzcd}
 & & & & K(i) \arrow[d, dash, "2"] \\
 & & & \exists \arrow[r] & E_k \arrow[d, dash, "2"] \\
K(i) \arrow[d, dash, "\text{power of } 2"'] & & & & \vdots \arrow[d, dash, "2"] \\
\mathbb{R} \arrow[rrr, squiggly, "\text{more Galois theory}"] & & & & E_1 \arrow[d, dash, "2"] & \text{impossible by (B)} \\
 & & & & \mathbb{C} \arrow[d, dash, "2"] & \text{unless } \mathbb{C} = K(i) \\
 & & & & \mathbb{R}
\end{tikzcd}$$

$\square$