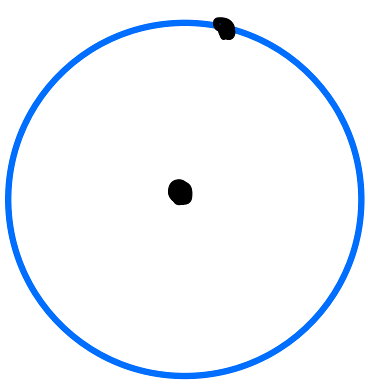
Straightedge and Compass Constructions

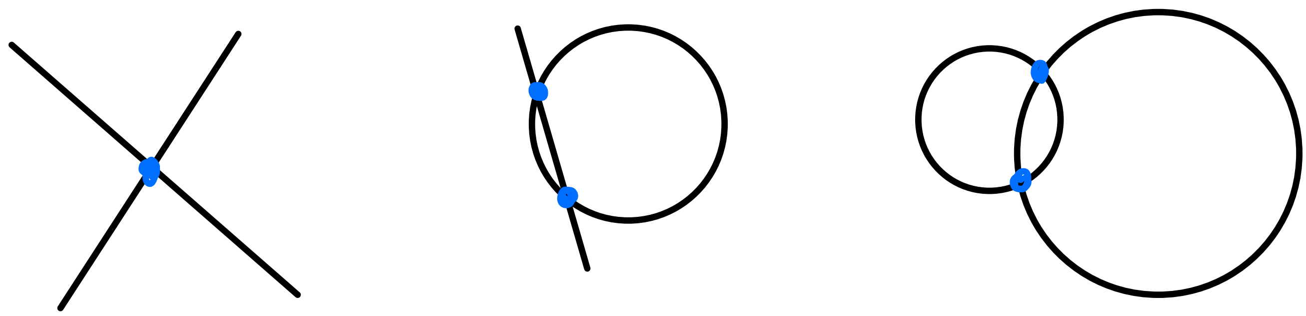
Operations:

1) Connect two pts. by a line

2) Draw a circle w/ a given center and point



3) Find int. pt. of lines/circles



3 problems that the Greeks couldn't solve:

I) "Double the cube"

II) Trisect an arbitrary angle

III) Square the circle"

Big idea: <u>constructible numbers</u>

Start w/ two points $\underset{0}{\cdot}$ $\underset{1}{\cdot}$

Constructible numbers:

$$\mathcal{C} := \left\{ z \in \mathbb{C} \;\middle|\; \begin{array}{c}\text{the pt. } z \text{ is constructible} \\ \text{from } 0 \text{ and } 1\end{array} \right\}$$

$$D := \{ d \in \mathbb{R} \mid \exists a, b \in \mathcal{C} \text{ w/ } |a-b| = d \}$$

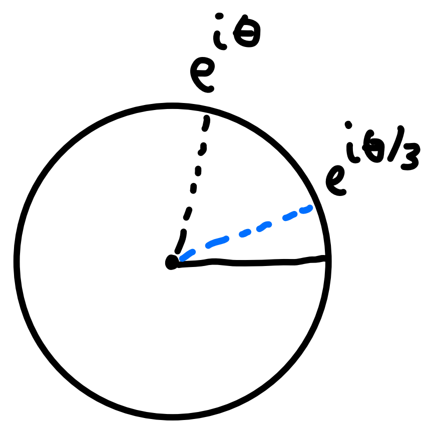
$$\mathcal{C}_{\mathbb{R}} := \mathcal{C} \cap \mathbb{R} \subseteq D$$

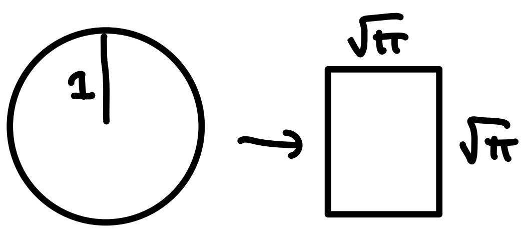
Rephrase:

I) Construct $\sqrt[3]{2}$

II) Construct $e^{i\theta/3}$ given $e^{i\theta}$

III) Construct $\sqrt{\pi}$





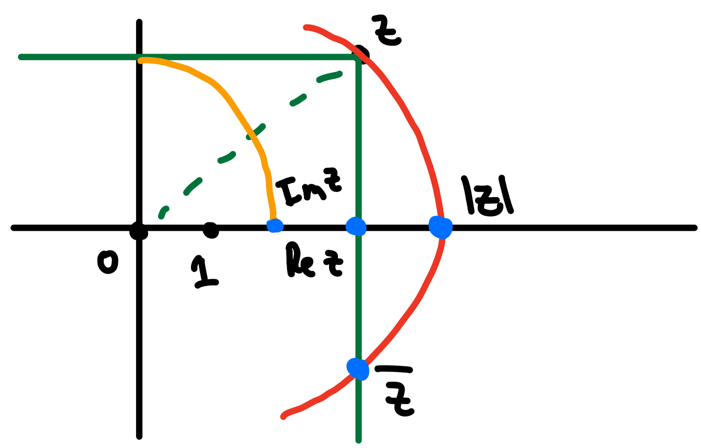
Prop: $\mathcal{C}$ is closed under

a) $z \mapsto |z|$

b) $z \mapsto \overline{z}$

c) $z \mapsto \mathrm{Re}(z)$

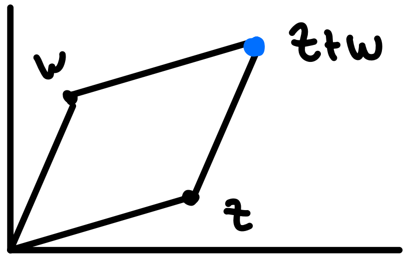
d) $z \mapsto \mathrm{Im}(z)$

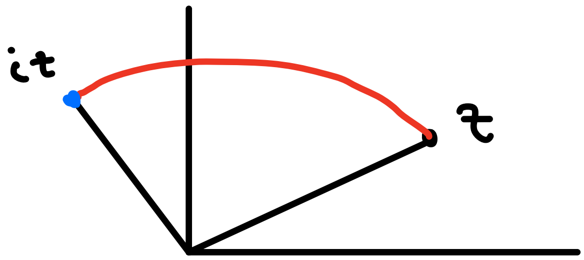


e) Addition

f) Subtraction

g) Mult by i





$\square$

Prop: $z = x + iy \in \mathcal{C} \iff x, y \in \mathcal{C}_{\mathbb{R}}$

Pf: $\Rightarrow$) c & d

$\Leftarrow$) e & g

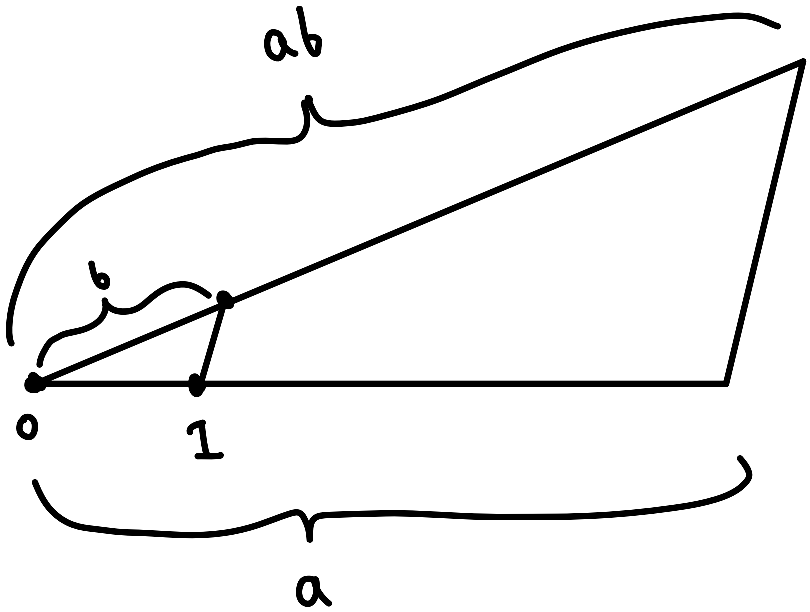
$\square$

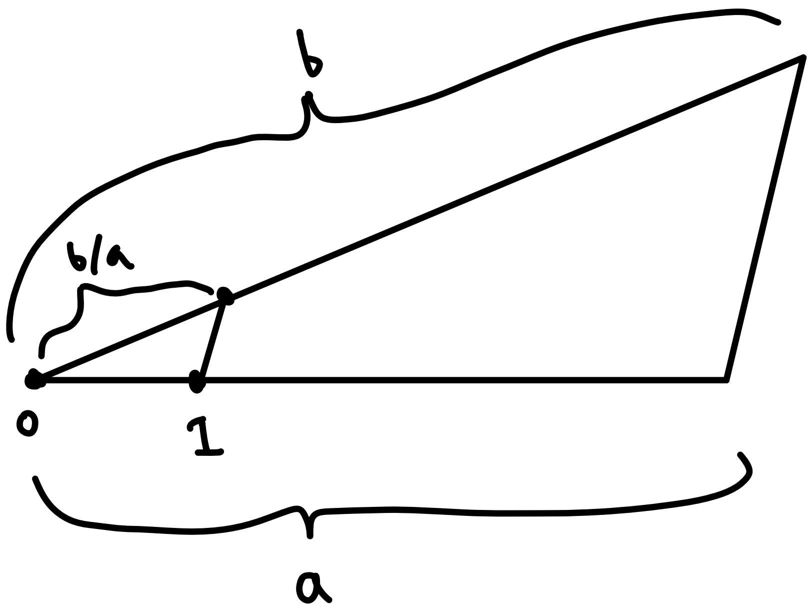
Prop: $D = \mathcal{C}_{\mathbb{R}}$

Pf: f & b

Prop: $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{C}$ are fields

Pf: Suffices to prove $\mathcal{C}_{\mathbb{R}}$ closed under mult. and division





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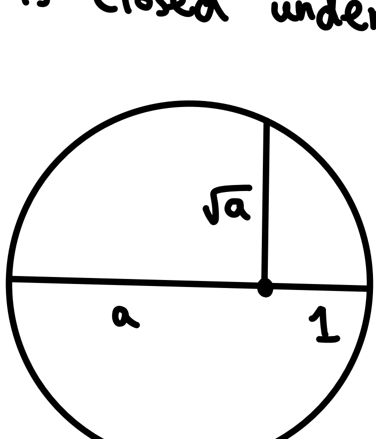
Prop: $\mathcal{C}_{\mathbb{R}}$ is closed under $\sqrt{\cdot}$

Pf:



$\square$

Thm: If $z \in \mathcal{C}$, then $[\mathbb{Q}(z) : \mathbb{Q}]$ is a power of 2.

Pf sketch: All intersections of lines / circles give quadratic eqns.

Cor:

I) Can't double the cube

II) Can't trisect an arbitrary angle

III) Can't square the circle

Pf:

I) Can't construct $\sqrt[3]{2}$ (min. poly: $x^3 - 2$)

II) Let $\theta = 60°$. Then $e^{i\theta} = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \in \mathcal{C}$, but $z = e^{i\theta/3}$ is a root of $x^6 - x^3 + 1$, which is irred. in $\mathbb{F}_2[x]$, and hence in $\mathbb{Q}[x]$.

III) Can't construct $\sqrt{\pi}$ since π and therefore $\sqrt{\pi}$ are transcendental

$\square$