Announcements

Midterm 1: Thurs. 2/15 7:00-8:30pm Loomis Lab. 144
Tomorrow's problem session & Wed. class: review
See Friday's email for full policies

Recall: Tower Law: If $F \subseteq K \subseteq L$,

$$[L:F] = [L:K][K:F]$$

Composite: Smallest subfield K_1K_2 of L containing K_1 and K_2.

Prop: Let K_1/F, K_2/F be finite ext'ns w/ $K_1, K_2 \in L$.

a) $[K_1K_2 : K_2] \leq [K_1 : F]$

b) $[K_1K_2 : F] \leq [K_1:F][K_2:F]$

Pf: Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis for K_1 over F.
Let $K = \{f_1\alpha_1 + \cdots + f_n\alpha_n \mid f_i \in K_2\}$
We have $K_1 \subseteq K$, $K_2 \subseteq K$, and $\dim_{K_2} K \leq n$, so if it's a field it is K_1K_2, and a) will hold.

Closed under $+, -$: yes, since K is a v.s.

Closed under $\cdot$:

Since $\alpha_1, \dots, \alpha_k$ is an F-basis for K_1, write

$$\alpha_i \alpha_j = \sum_k \underset{\in F \subseteq K_2}{h_k} \alpha_k$$

Then,

$$(f_1\alpha_1 + \dots + \overset{\in K_2}{f_n}\alpha_n)(g_1\alpha_1 + \dots + \overset{\in K_2}{g_n}\alpha_n)$$

$$= \sum_{i,j,k} \underbrace{f_i g_j}_{\in K_2} \underbrace{\alpha_i \alpha_j}_{\in K_1} = \sum_{i,j,k} f_i g_j h_k \alpha_k = \sum_k \underbrace{\left(\sum_{i,j} f_i g_j h_k\right)}_{\in K_2} \alpha_k$$

Inverses: Let $\gamma \in K \setminus \{0\}$, and consider the K_2-linear transformation

$$T_\gamma : K \longrightarrow K \qquad \left(\begin{array}{l}\text{additive gp. homom.,}\\ \text{but } \underline{\text{not}} \text{ ring homom.}\end{array}\right)$$

$$a \longmapsto a\gamma$$

Since L is an integral domain,

$\ker(T_\gamma) = \{0\}$, so by the rank-nullity theorem,

$$\dim \operatorname{im} T_\gamma + \underbrace{\dim \ker T_\gamma}_{0} = n, \text{ so } T_\gamma \text{ is onto.}$$

Thus γ has inverse $T_\gamma^{-1}(1) \in K$.

b) Using the Tower Law,

$$[K_1:F][K_2:F] \geq [K_1K_2:K_2][K_2:F] = [K_1K_2:F]$$

$\square$

Alternate pf (see D&F): Finite ext'ns are interated simple extensions. Prove a) for simple ext'ns by considering degrees of min'l polys, and use induction for the general case

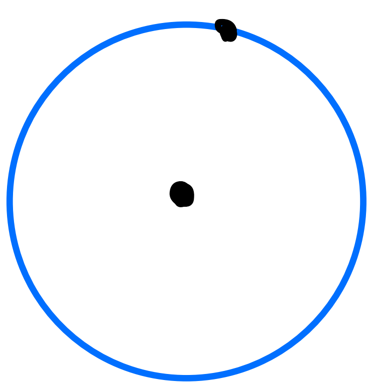
Straightedge and Compass Constructions

Game (ancient Greeks): Given a straightedge (ruler w/out markings) and compass, what can we construct?

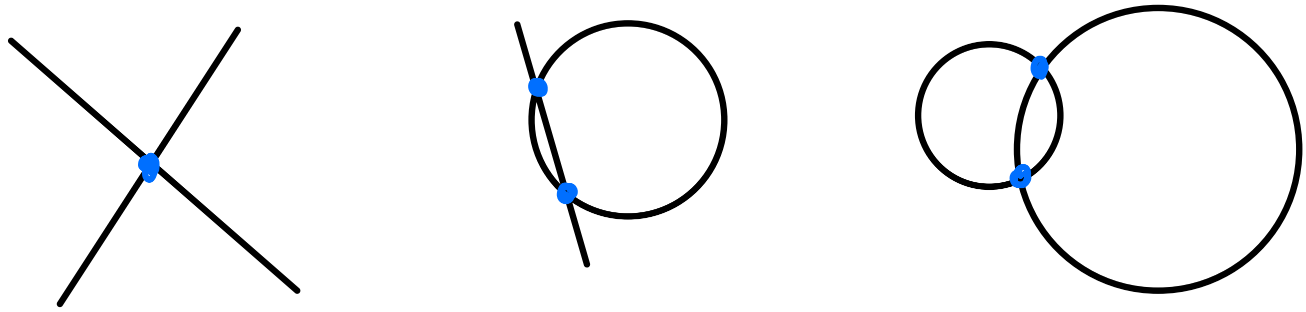
Operations:

1) Connect two pts. by a line

2) Draw a circle w/ a given center and point



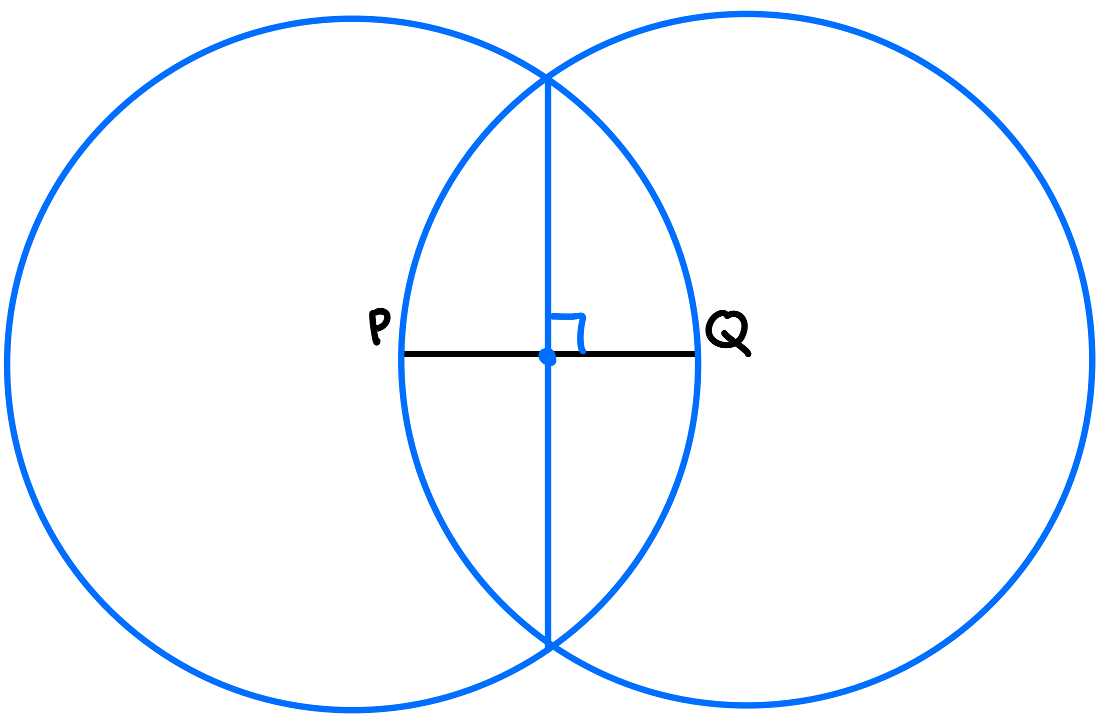
3) Find int. pt. of lines/circles



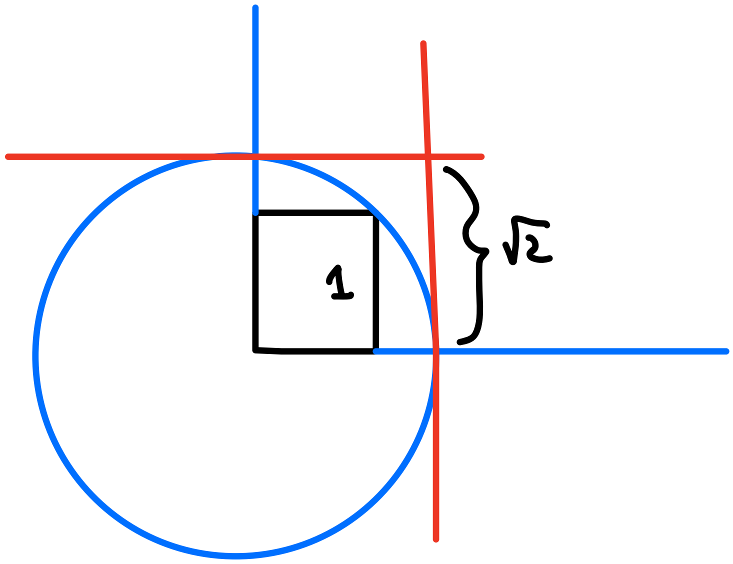
No measuring allowed!

With these operations, can do many things:

a) Perpendicular bisector



b) Double the area of a square



c) Construct the n-gon for <u>certain</u> n

(Gauss: 17-gon)

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3 problems that the Greeks couldn't solve:

I) "Double the cube"

II) Trisect an arbitrary angle

III) Square the circle"

Big idea: <u>constructible numbers</u>

Start w/ two points $\cdot$ 0 $\cdot$ 1

Constructible numbers:

$$\mathcal{C} := \left\{ z \in \mathbb{C} \;\middle|\; \begin{array}{c}\text{the pt. } z \text{ is constructible} \\ \text{from } 0 \text{ and } 1\end{array} \right\}$$

Rephrase:

I) Construct $\sqrt[3]{2}$

II) Construct $\cos\frac{\theta}{3}$ given $\cos\theta$

III) Construct $\sqrt{\pi}$