

Math 418, Spring 2024 – Homework 6

Due: Wednesday, March 6th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition*. All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 2 bonus points per assignment.

1. **Dummit and Foote #13.5.3:** Prove that d divides n if and only if $x^d - 1$ divides $x^n - 1$. (Hint: if $n = qd + r$, then $x^n - 1 = (x^{qd+r} - x^r) + (x^r - 1)$)
2. **Dummit and Foote #13.5.6:** Prove that $x^{p^n-1} - 1 = \prod_{\alpha \in \mathbb{F}_{p^n}^\times} (x - \alpha)$. Conclude that $\prod_{\alpha \in \mathbb{F}_{p^n}^\times} \alpha = (-1)^{p^n}$ so the product of the nonzero elements of a finite field is $+1$ if $p = 2$ and -1 if p is odd. For p odd and $n = 1$ derive Wilson's Theorem: $(p - 1)! = -1 \pmod{p}$.
3. **Dummit and Foote #13.6.2:** Let ζ_n be a primitive n th root of unity and let d be a divisor of n . Prove that ζ_n^d is a primitive (n/d) th root of unity.
4. **Dummit and Foote #13.6.3:** Prove that if a field contains the n th roots of unity for n odd then it also contains the $2n$ th roots of unity.
5. **Dummit and Foote #13.6.7:** Use the Mobius Inversion formula indicated in Section 14.3 to prove

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}.$$

6. **Dummit and Foote #14.1.3:** Determine the fixed field of complex conjugation on \mathbb{C} .
7. **Dummit and Foote #14.1.5:** Determine the automorphisms of the extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ explicitly. (Hint: Use Dummit & Foote Proposition 14.5)