

# Math 418, Spring 2024 – Homework 3

**Due:** Wednesday, February 7th, at 9:00am via Gradescope.

**Instructions:** Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition*. All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 2 bonus points per assignment.

1. **Dummit and Foote #9.3.2:** Prove that if  $f(x)$  and  $g(x)$  are polynomials with rational coefficients whose product  $f(x)g(x)$  has integer coefficients, then the product of any coefficient of  $g(x)$  with any coefficient of  $f(x)$  is an integer.
2. **Dummit and Foote #9.4.2d:** Let  $p$  be an odd prime. Prove that the polynomial  $f(x) = \frac{(x+2)^p - 2^p}{x}$  is irreducible in  $\mathbb{Z}[x]$ .
3. **Dummit and Foote #9.4.10:** Prove that the polynomial  $p(x) = x^4 - 4x^2 + 8x + 2$  is irreducible over the quadratic field  $F = \mathbb{Q}(\sqrt{-2}) = \{a + b\sqrt{-2} | a, b \in \mathbb{Q}\}$ .
4. **Dummit and Foote #9.4.12:** Prove that  $f(x) = x^{n-1} + x^{n-2} + \cdots + x + 1$  is irreducible over  $\mathbb{Z}$  if and only if  $n$  is a prime.
5. **Dummit and Foote #13.1.1:** Show that  $p(x) = x^3 + 9x + 6$  is irreducible in  $\mathbb{Q}[x]$ . Let  $\theta$  be a root of  $p(x)$ . Find the inverse of  $1 + \theta$  in  $\mathbb{Q}(\theta)$  as a polynomial in  $\theta$ .
6. **Dummit and Foote #13.1.3:** Show that  $p(x) = x^3 + x + 1$  is irreducible over  $\mathbb{F}_2$  and let  $\theta$  be a root. Compute the powers of  $\theta$  in  $\mathbb{F}_2(\theta)$  as polynomials in  $\theta$  of degree  $\leq 2$ .
7. **Dummit and Foote #13.1.4:** Prove directly that the map  $a + b\sqrt{2} \mapsto a - b\sqrt{2}$  is an isomorphism of  $\mathbb{Q}(\sqrt{2})$  with itself.