Math 418, Spring 2024 – Homework 10

Due: Wednesday, April 31st, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 2 bonus points per assignment.

1. Let k be an algebraically closed field, and consider the polynomial ring $k[x, y]$.
   (a) Let V be the x-axis, i.e. $V = V(y)$. Prove that V is irreducible. [Hint: Show a prime ideal is radical.]
   (b) Prove that $V = V(x - y)$ is irreducible.
   (c) Prove that $S = \{(a, a) \in k^2 | a \neq 1\}$ is *not* an algebraic variety if $k = \mathbb{C}$.
   (d) What is the decomposition of $V = V(x^2 - y^2)$ into irreducibles? **Warning:** The answer depends on k!

2. **Dummit and Foote #15.1.2** Show that each of the following rings are not Noetherian by exhibiting an explicit infinite increasing chain of ideals:
   (a) the ring of continuous real valued functions on $[0, 1]$,
   (b) the ring of all functions from any infinite set X to $\mathbb{Z}/2\mathbb{Z}$.

3. **Dummit and Foote #15.1.20** *If f and g are irreducible polynomials in $k[x, y]$ that are not associates (do not divide each other), show that $V((f, g))$ is either $\emptyset$ or a finite set in k^2.* [Hint: If $(f, g) \neq (1)$, show (f, g) contains a nonzero polynomial in $k[x]$ (and similarly a nonzero polynomial in $k[y]$) by letting $R = k[x], F = k(x)$, and applying Gauss's Lemma to show f and g are relatively prime in $F[y]$.]

4. **Dummit and Foote #15.2.2** Let I and J be ideals in the ring R. Prove the following statements:
   (a) If $I^k \subseteq J$ for some $k \geq 1$, then $\sqrt{I} \subseteq \sqrt{J}$.
   (b) If $I^k \subseteq J \subseteq I$ for some $k \geq 1$, then $\sqrt{I} = \sqrt{J}$.
   (c) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
   (d) $\sqrt{\sqrt{I}} = \sqrt{I}$.

(e) $\sqrt{I}+\sqrt{J} \subseteq \sqrt{I+J}$ and $\sqrt{I+J} = \sqrt{\sqrt{I}+\sqrt{J}}$.

5. **Dummit and Foote #15.2.3** *Prove that the intersection of two radical ideals is again a radical ideal.*

6. **Dummit and Foote #15.2.5** *If $I = (xy, (x-y)z) \subseteq k[x, y, z]$ prove that $\sqrt{I} = (xy, xz, yz)$. For this ideal prove* directly *that $V(I) = V(\sqrt{I})$, that $V(I)$ is not irreducible, and that $\sqrt{I}$ is not prime.*