Math 418, Spring 2024 – Homework 10

Due: Wednesday, April 31st, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 2 bonus points per assignment.

1. Let k be an algebraically closed field, and consider the polynomial ring $k[x, y]$.

   (a) Let V be the x-axis, i.e. $V = V(y)$. Prove that V is irreducible. [Hint: Show a prime ideal is radical.]

   Solution. If I is a prime ideal, then if $a \cdots a = a^n \in I$, then a or a or ... or a is in I, so $a \in I$ and so I is radical.

   y is irreducible since it is degree 1, so it is prime since $k[x, y]$ is a UFD. Therefore, (y) is a prime ideal, so by the bijection proved in class, $V(y)$ is irreducible.

   (b) Prove that $V = V(x - y)$ is irreducible.

   Solution. Similarly, $x - y$ is irreducible since it is degree 1, so it is prime since $k[x, y]$ is a UFD. Therefore, $(x - y)$ is a prime ideal, so by the bijection proved in class, $V(x - y)$ is irreducible.

   (c) Prove that $S = \{(a, a) \in k^2 | a \neq 1\}$ is *not* an algebraic variety if $k = \mathbb{C}$.

   Solution. Let $V = \{(a, a) \in k^2 | a \in k\}$. then this is a variety with $I(V) = (x-y)$. If S is a variety, we have $V(I(S)) = S$. Let $f(x, y) \in I(S)$, and let $g(x) = f(x, x)$. Since $f \in I(V)$, $f(a, a) = 0$ for all $a \neq 0$, so $g(x)$ has roots at all $a \neq 0$. This is infinitely many roots and g is a polynomial, so $g = 0$, and so $f(0, 0) = 0$, so $f \in I(V)$. This means that $I(S) = I(V)$, and $V(I(S)) \neq S$, meaning that S is not a variety.

   (d) What is the decomposition of $V = V(x^2 - y^2)$ into irreducibles? **Warning:** The answer depends on k! **Solution.** We have $x^2 - y^2 = (x + y)(x - y)$, so we have $V = V(x + y) \cup V(x - y)$. However, if char $k = 2$, then $x + y = x - y$, so in that case we simply have $V = V(x + y)$.

2. **Dummit and Foote #15.1.2** Show that each of the following rings are not Noetherian by exhibiting an explicit infinite increasing chain of ideals:

   (a) the ring of continuous real valued functions on $[0, 1]$

Solution. Let

$$I_j = \{\text{functions } f : \mathbb{R} \to [0,1] | f(x) = 0 \text{ for all } |x| > j\}.$$

A direct check shows that this is an ideal, and we have

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots .$$

(b) the ring of all functions from any infinite set X to $\mathbb{Z}/2\mathbb{Z}$.
Solution. Let $a_1, a_2, \ldots$ be distinct elements of X. Let

$$I_j = \{\text{functions } f : X \to \mathbb{Z}/2\mathbb{Z} | f(a_i) = 0 \text{ for all } i \geq j\}.$$

A direct check shows that this is an ideal, and we have

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots .$$

3. **Dummit and Foote #15.1.20** *If f and g are irreducible polynomials in $k[x, y]$ that are not associates (do not divide each other), show that $V((f, g))$ is either $\emptyset$ or a finite set in k^2.* [Hint: If $(f, g) \neq (1)$, show (f, g) contains a nonzero polynomial in $k[x]$ (and similarly a nonzero polynomial in $k[y]$) by letting $R = k[x], F = k(x)$, and applying Gauss's Lemma to show f and g are relatively prime in $F[y]$.]

   Solution. Use the hint. If $(f, g) = (1)$, then $V((f, g)) = \emptyset$. Otherwise, let $R = k[x], F = k(x)$, and consider f and g as elements of both $R[y] = k[x, y]$ and $F[y]$. R is a UFD, so by Gauss' lemma, f and g are irreducible in $F[y]$ since they are irreducible in $R[y]$. Since f and g are irreducible nonassociates, they are relatively prime, and since $F[y]$ is a Euclidean domain we have $fa + gb = 1$ for some $a, b \in F[y]$. Multiplying by a large enough power of x, we obtain $f\tilde{a} + g\tilde{b} \in k[x]$ for some $\tilde{a}, \tilde{b} \in k[x, y]$; in other words (f, g) contains an element $p \in k[x]$. By a similar argument, (f, g) contains an element $q \in k[y]$. Every element of $V((f, g))$ must be a root of p and q, so must be of the form (a, b) with $p(a) = q(b) = 0$. Since p and q are one-variable polynomials, they have finitely many roots, so $V((f, g))$ is finite.

4. **Dummit and Foote #15.2.2** Let I and J be ideals in the ring R. Prove the following statements:

   (a) If $I^k \subseteq J$ for some $k \geq 1$, then $\sqrt{I} \subseteq \sqrt{J}$.
   Solution. If $x \in \sqrt{I}$, $x^n \in I$ for some n, so since $I^k \subseteq J$, $x^{kn} \in J$. Therefore, $x \in \sqrt{J}$.

   (b) If $I^k \subseteq J \subseteq I$ for some $k \geq 1$, then $\sqrt{I} = \sqrt{J}$.
   Solution. Applying the previous part twice, we have $\sqrt{I} \subseteq \sqrt{J} \subseteq \sqrt{I}$, so $\sqrt{I} = \sqrt{J}$.

(c) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Solution. If $x \in \sqrt{IJ}$, $x^n \in IJ \subseteq I \cap J$ for some n, so $x \in \sqrt{I \cap J}$. If $y \in \sqrt{I \cap J}$, $y^n \in I \cap J$ for some n, so $y^n \in I$ and $y^n \in J$, and $x \in \sqrt{I} \cap \sqrt{J}$. If $z \in \sqrt{I} \cap \sqrt{J}$, then for some m, n, $z^m \in I, z^n \in J$. Therefore, $z^{m+n} \in IJ$, so $z \in \sqrt{IJ}$.

(d) $\sqrt{\sqrt{I}} = \sqrt{I}$.

Solution. If $x \in \sqrt{\sqrt{I}}$, for some m, $x^m \in \sqrt{I}$, so for some n, $x^{mn} = (x^m)^n \in I$, so $x \in \sqrt{I}$. Conversely, every ideal is contained in its radical.

(e) $\sqrt{I} + \sqrt{J} \subseteq \sqrt{I+J}$ and $\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

Solution. If $z \in \sqrt{I} + \sqrt{J}$, we have $z = x + y$ with $x \in \sqrt{I}, y \in \sqrt{J}$. For some m, n, $x^m \in I, y^n \in J$, so $(x+y)^{m+n} = x^{m+n} + x^{m+n-1}y + \cdots + x^m y^n + \cdots + y^{m+n} \in I + J$ since every term has a factor of x^m or y^n. Thus, $\sqrt{I} + \sqrt{J} \subseteq \sqrt{I+J}$. By part a, $\sqrt{\sqrt{I} + \sqrt{J}} \subseteq \sqrt{I+J}$. Conversely, $I \subseteq \sqrt{I}, J \subseteq \sqrt{J}$, so $I + J \subseteq \sqrt{I} + \sqrt{J}$ and $\sqrt{I+J} \subseteq \sqrt{\sqrt{I} + \sqrt{J}}$.

5. **Dummit and Foote #15.2.3** *Prove that the intersection of two radical ideals is again a radical ideal.*

Solution. Let I and J be radical ideals, and let $x^n \in I \cap J$. Then $x^n \in I$, so since I is radical, $x \in I$. Similarly, $x^n \in J$, so since J is radical $x \in J$. Therefore, $x \in I \cap J$, so $I \cap J$ is radical.

6. **Dummit and Foote #15.2.5** *If $I = (xy, (x-y)z) \subseteq k[x, y, z]$ prove that $\sqrt{I} = (xy, xz, yz)$. For this ideal prove* directly *that $V(I) = V(\sqrt{I})$, that $V(I)$ is not irreducible, and that $\sqrt{I}$ is not prime.*

Solution. $z^2 \cdot xy + xz(x-y)z = x^2z^2 \in I$, so $xz \in \sqrt{I}$. Since $xy, (x-y)z \in I \subseteq \sqrt{I}$, $yz = xz - (x-y)z \in \sqrt{I}$. Now, (xy, xz, yz) contains all monomials with more than one variable, so since $x^n, y^n, z^n \notin I$ for any n, none of them are in $\sqrt{I}$ either, so $\sqrt{I} = (xy, xz, yz)$.

We know that $V(I) = V(\sqrt{I})$ by the Nullstellensatz, but the problem asks to show it directly. $a = (x, y, z) \in V(\sqrt{I})$ iff $xy = xz = yz = 0$ iff at least two of x, y, and z are 0. On the other hand, $a = (x, y, z) \in V(I)$ iff $xy = 0$ and $(x-y)z = 0$ iff either $x = 0$ and $-yz = 0$ or $y = 0$ and $xz = 0$ iff at least two of x, y, and z are 0.

$V(I) = \{(x, 0, 0)\} \cup \{(0, y, 0)\} \cup \{(0, 0, z)\} = V((y, z)) \cup V((x, z)) \cup V((x, y))$, and since none of these varieties is contained in the others, $V(I)$ is reducible. Finally, $\sqrt{I}$ is not prime since $xy \in \sqrt{I}$ but $x, y \notin \sqrt{I}$.