Math 418, Spring 2024 – Homework 1

Due: Wednesday, January 24th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 2 bonus points per assignment.

1. Dummit and Foote #7.1.3
2. Dummit and Foote #7.1.11
3. Dummit and Foote #7.2.1
4. Dummit and Foote #7.3.2
5. Dummit and Foote #7.4.15
6. Consider $R = \mathbb{Z}[\sqrt{-5}]$ with the (non-Euclidean) norm $N : R \to \mathbb{Z}_{\geq 0}$ given by $N(a) = |a|^2$. Note that $N(a \cdot b) = N(a)N(b)$.
   (a) Prove that $a \in R$ is a unit if and only if $N(a) = 1$. Find all the units in R.
   (b) Recall that $r \in R$ is irreducible if whenever $r = ab$ then one of a or b is a unit. Use the norm to show that 2, 3, $1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are all irreducible elements of R
   (c) Show that $2, 3, 1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are not unit multiples of one another, proving that R lacks unique factorization since $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.
7. Let R be an integral domain. Recall that g is a greatest common divisor of two elements $a, b \in R$ if g divides a and b, and if d divides a and b then d divides g.
   (a) Show that if g and g' are two gcds of $a, b \in R$, $g' = ug$ for some unit u.
   (b) Let $R = \mathbb{Z}[\sqrt{-5}]$. Prove that 6 and $2 + 2\sqrt{-5}$ have no gcd. *(Hint: Use the fact that 2 and $1 + \sqrt{-5}$ are both common divisors of these elements)*