Final exam: Thurs 12/14, 8:00-11:00 am, 132 Berier Hall

TWO reference sheets (2x front and back) allowed

Cumulative: everything from the course is fair game

See email for full policies

Today: ~60 minutes of prepared problems

then anything you want to talk about

Examples:

1) (Rem. 4.3.13) Let D be a digraph and let $x,y \in V(D)$. Use network flows to prove that $\kappa'(x,y) = \lambda'(x,y)$.

$\kappa'(x,y)$ = size of a minimum x,y-edge cut

$\lambda'(x,y)$ = maximum num. of edge-disjoint x,y-paths

Remark: This is one of 4 versions of Menger's Thm.

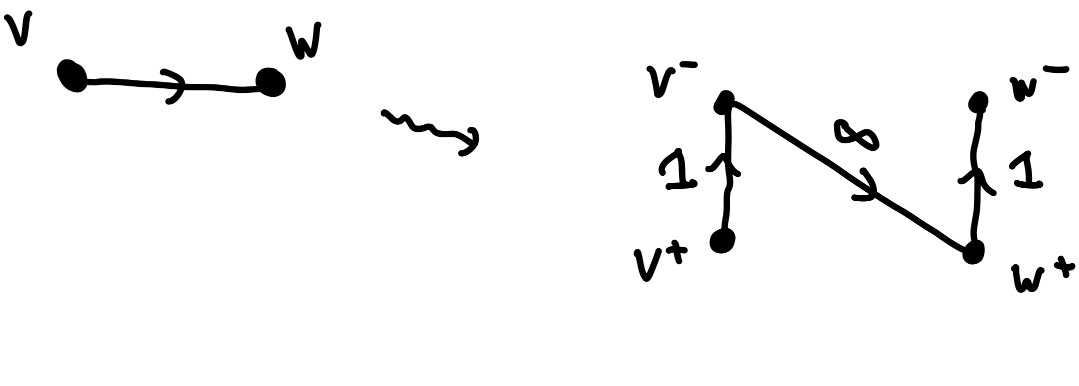
Pf: If $\exists k$ edge-disjoint x,y-paths, then every x,y-edge cut must contain ≥ 1 edge from every path, so has $\geq k$ edges. Thus, $\kappa'(x,y) \geq \lambda'(x,y)$.

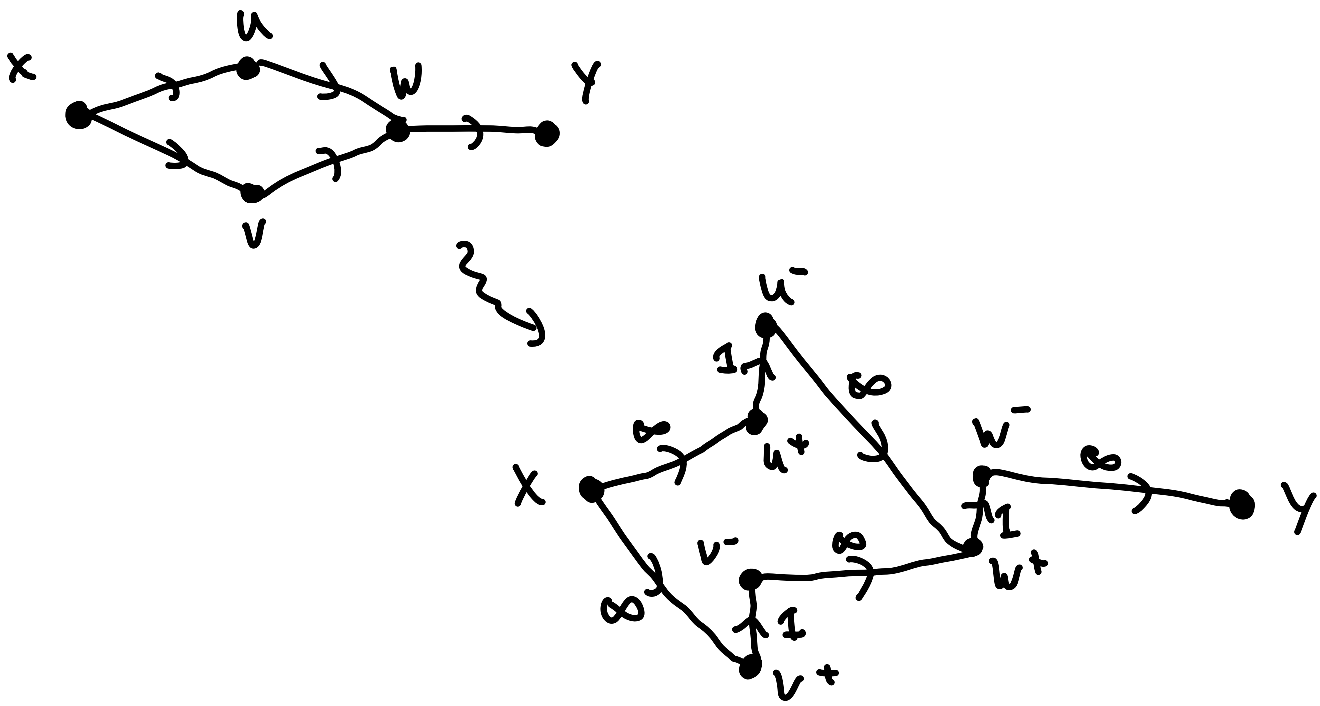
Conversely, consider D as a network w/ source

x and sink y, where every edge has capacity 1.

By the integrality theorem (4.3.12), there is a maximum flow f where all edge flows are integers. Let k be value of this flow; f then determines a set of k x,y-paths and since every edge flow is 0 or 1, these paths are edge disjoint. Thus $\lambda'(x,y) \geq k$.

By the max.-flow, min.-cut thm., $\exists$ a (minimum) edge cut $[S,T]$ w/ capacity k. Since every edge has capacity 1, this means that $[S,T]$ consists of k edges, and since $[S,T]$ is an x,y-edge cut, we have $\kappa'(x,y) \leq k$. Combining these facts yields $\lambda'(x,y) \geq \kappa'(x,y)$, so $\lambda'(x,y) = \kappa'(x,y)$. □

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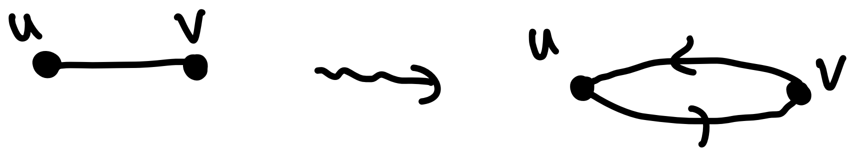
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2) (2.1.32) Let G: conn. graph, $e \in E(G)$. Prove that

a) e is a cut-edge $\Leftrightarrow$ e belongs to every spanning tree

b) e is a loop $\Leftrightarrow$ e belongs to no spanning tree

Pf: a) e is a cut-edge $\Leftrightarrow$ $G \setminus e$ is disconn.

$\Leftrightarrow$ $T \setminus e$ is disconn. for every spanning tree T of G

WTS: $G \setminus e$ is conn. $\Rightarrow$ $T \setminus e$ is conn. for some spanning tree T of G

Pf: Take a spanning tree T of $G \setminus e$.
Then $T \setminus e = T$ is conn.

b) $\Rightarrow$) Trees have no cycles, so no loops.

$\Leftarrow$) If e is not a loop, let T be a spanning tree of G. Either $e \in E(T)$ or $T \cup e$ contains a cycle. Let $f \in E(T)$ be another edge in that cycle (such an edge exists since e is not a loop). Then, $(T \cup e) \setminus f$ is conn. and has the same number of edges as T, so is a spanning tree containing e. $\square$

3) (3.1.29) a) Prove that every bipartite graph $\hat{G}$ has a matching of size $\geq \frac{e(G)}{\Delta(G)}$

b) Let H be a subgraph of $K_{n,n}$ w/ $> (k-1)n$ edges. Prove that H has a matching of size $\geq k$.

Pf: a) Recall the König-Egerváry Thm: $\alpha'(G) = \beta(G)$ for all bipartite graphs where

$\alpha'(G)$ = maximum size of matching

$\beta(G)$ = minimum size of vertex cover

Every vertex in a vertex cover must cover $\leq \Delta(G)$ edges, so the vertex cover has $\geq e(G)/\Delta(G)$ vertices. By König-Egerváry, $\alpha'(G) = \beta(G) \geq e(G)/\Delta(G)$.

$U \subseteq V(G)$ $\quad U$: vertex cover

$\forall e \in E(G)$, ≥ 1 endpoint of e is in U

$u \in U$ is an endpoint of $\leq \Delta(G)$ edges $e \in E(G)$.

$U \subseteq V(G)$ $\quad U = \{u_1, \ldots, u_k\}$

Q: how many edges are "covered" by U?

i.e. how many edges have ≥ 1 endpoint in U?

this number is

$$a \leq d(u_1) + d(u_2) + \cdots + d(u_k) \leq \underbrace{\Delta(G) + \cdots + \Delta(G)}_{k \text{ terms}}$$

$$= k\Delta(G)$$

If U is a vertex cover, $a = e(G)$, so

$$e(G) \leq k\Delta(G) = |U|\,\Delta(G), \text{ so}$$

$$|U| \geq \frac{e(G)}{\Delta(G)}$$

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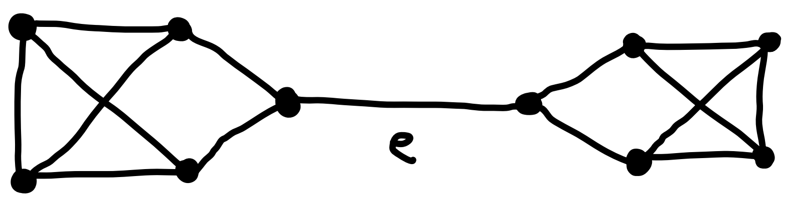
b) For a subgraph H of $K_{n,n}$, $\Delta(H) \leq n$, and by assumption, $e(H) > (k-1)n$. By part a,

$$\alpha'(H) \geq e(H)/\Delta(H) > \frac{(k-1)n}{\Delta(H)} \geq \frac{(k-1)n}{n} = k-1,$$

So $\alpha(H) \geq k$. $\square$

4) (4.1.10): Determine the smallest 3-regular simple graph G with $\underbrace{\kappa(G)=1}_{\text{(vertex) connectivity}}$

Pf: By Theorem 4.1.11, $\kappa(G)=\kappa'(G)$ for 3-regular simple graphs, so we find the smallest 3-regular simple graph G w/ $\kappa'(G)=1$.



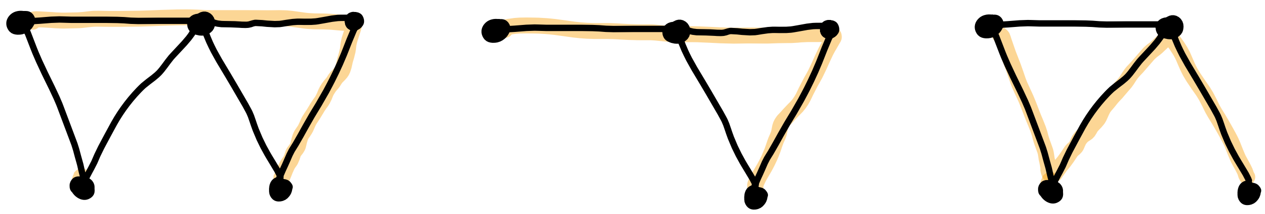
[10 vertices]

Let G be a 3-reg. simple graph w/ $\kappa'(G)=1$. Let H be a component of $G \setminus e$ for some cut edge e. Since H has a vertex of deg. 3, $n(H) \geq 4$. By the deg. sum formula,

$$2e(H) = 2 + 3(n(H)-1),$$

so $n(H)-1$ is even, so $n(H)$ is odd, and so $n(H) \geq 5$. Thus, $n(G) \geq 10$. $\square$

5) Let G be a P_4-free simple graph (no <u>induced</u> subgraph isom. to P_4). Prove that the greedy coloring algorithm uses $\chi(G)$ colors for <u>any</u> vertex order.

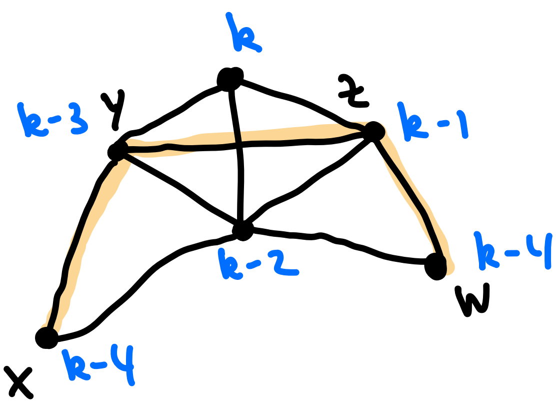


P_4-free

Pf: Let $v_1, \dots, v_n$ be an ordering of $V(G)$, and suppose the greedy coloring alg. uses k colors w.r.t. this ordering. We show that $\omega(G) \geq k$, so $\chi(G) = k$ since $\omega(G) \leq \chi(G) \leq k \leq \omega(G)$.

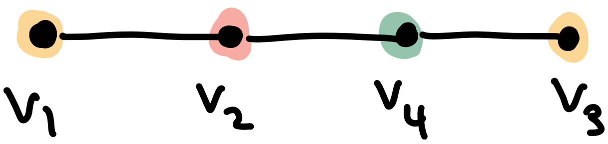
Let f be the proper coloring obtained via the greedy alg. and let i be the smallest integer s.t. G has a clique $Q = \{u_i, \dots, u_k\}$ of vertices colored $i, i+1, \dots, k$. (Clearly, $i \leq k$ since G has a vertex of color k)

If $i > 1$, by the greedy alg., each elt. of Q has a neighbor of color $i-1$. Let x be a vertex of color $i-1$ that is adjacent to the most vertices of Q. By the choice of i, $\exists z \in Q \setminus N(x)$. Let $w \in N(z)$ have color $i-1$; by the choice of x, $|N(x) \cap Q| \geq |N(w) \cap Q|$, so $\exists y \in (N(x) \cap Q) \setminus N(w)$. Since x and w have the same color, they are nonadjacent. To summarize,



x is adjacent to y, but not to z or w
y is adj. to x and z, but not to w
z is adj. to y and w, but not to x

So $V[\{x,y,z,w\}] \cong P_4$. Therefore, if G is P_4-free, $i=1$, so $|Q|=k$, $\omega(G) \geq k$.



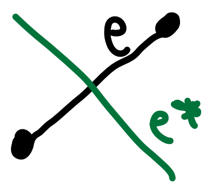
$G \longleftrightarrow G^*$

dual graph

$e(G) = e(G^*)$

bij. $e \in E(G) \leftrightarrow e^* \in E(G^*)$

$$2e(G^*) = \sum_{v \in V(G^*)} d(v)$$

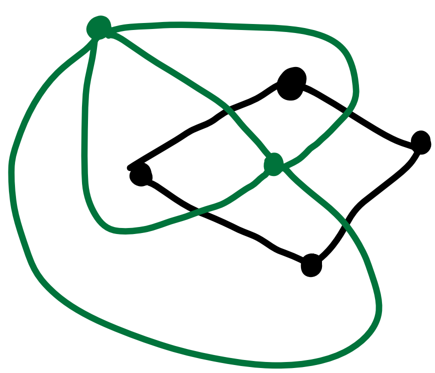


$$2e(G) = \sum_{f \text{: face of } G} \ell(f)$$

bij.

$v \in V(G^*) \leftrightarrow f$ face of G

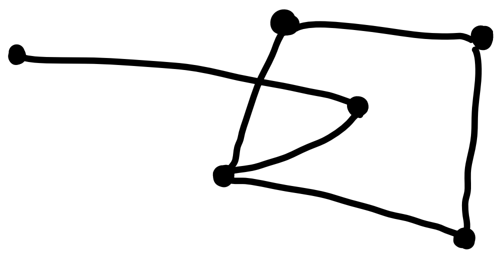
$$2e(G) = \sum_{v \in V(G)} d(v)$$



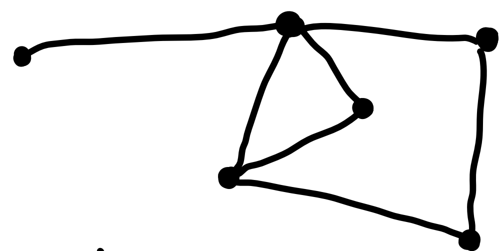
$d(v) = \ell(f)$

length

for us: Jordan curve theorem means that any closed curve (i.e. the boundary of a face of a plane graph) has an inside and an outside, and the only way to get from the inside to the outside is to cross the curve.



not planar



path goes thru. vertex in bdy. cycle

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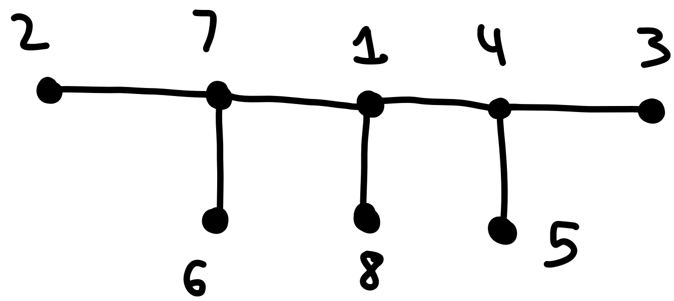

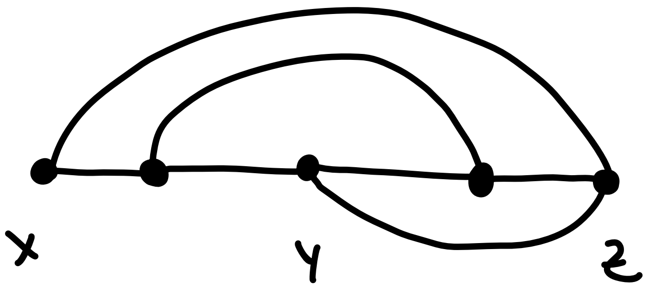
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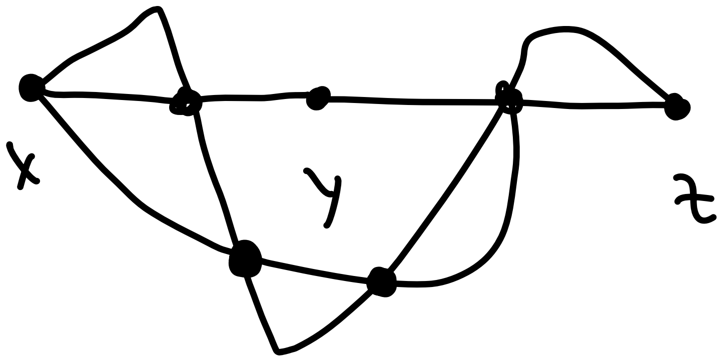
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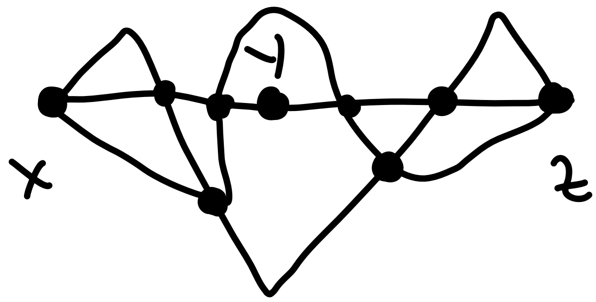
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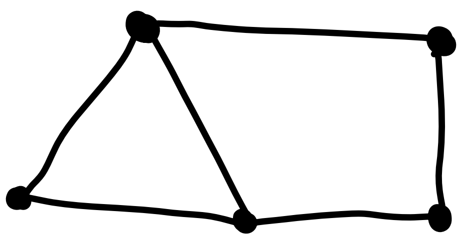

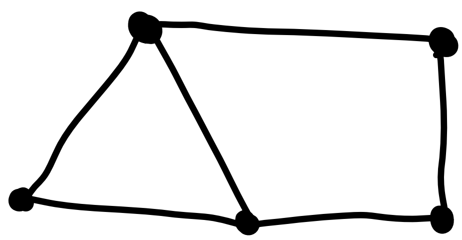

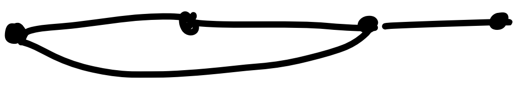

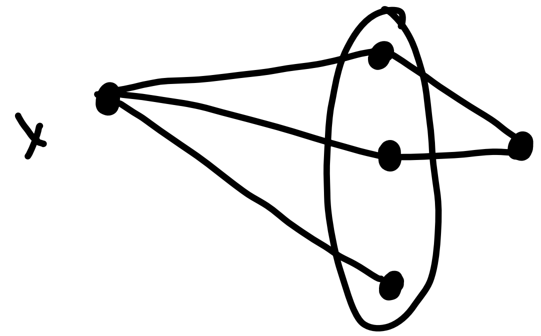

$N(x)$

$N(x)$ indep. set $\Rightarrow$ every edge has ≥ 1 endpoint <u>not</u> in $N(x)$

$$\sum_{v \notin N(x)} d(v) = \sum_{e \in E(G)} (\#\text{ endpoints of } e \text{ not in } N(x))$$

$$= \sum_{e \in E(G)} (1 \text{ or } 2)$$

$$\geq \sum_{e \in E(G)} 1 = e(G)$$