Announcements:

Quiz 1 this Friday in class (20 mins; start in middle)

Content: anything covered thru. Wednesday

Focus on definitions, thm. statements, examples

No outside resources allowed

E.g. "State the Havel-Hakimi Theorem, and give d and d' for the following graph: ..."

Midterm 1: Wed. 9/20 7:00-8:30pm (Noyes Lab. 217)

Will (roughly) cover through this week

Harder than quiz, more like homework

Accomodations/conflicts: contact me ASAP!

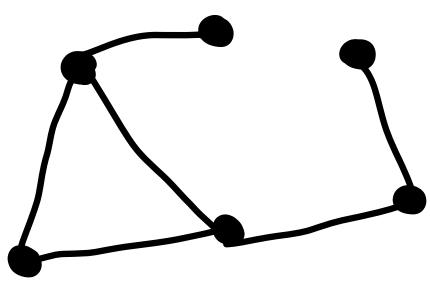
I will send a full email with policies soon

Havel-Hakimi Theorem:

a) For 1 vertex, the only graphic sequence is $d_1 = 0$

b) A list d of $n>1$ integers is graphic iff d' is graphic, where d' is obtained by deleting the largest element Δ and subtracting 1 from its next Δ largest elements

Ex:



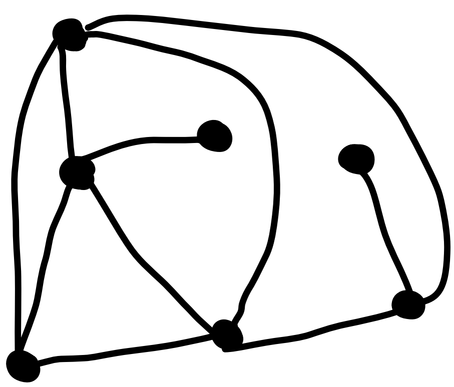
3,3,2,2,1,1 is graphic

So 4,4,4,3,3,1,1 is graphic

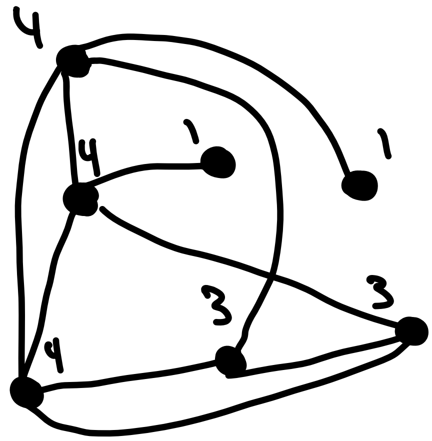
Since $\cancel{4}$, 4, 4, 3, 3, 1, 1

-1 -1 -1 -1

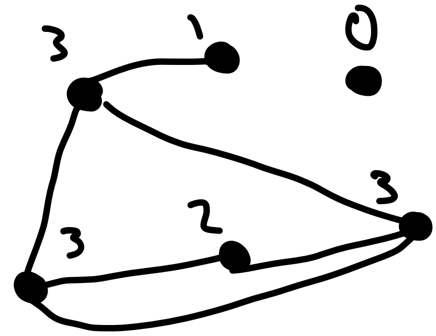
3, 3, 2, 2, 1, 1



4, 4, 4, 3, 3, 1, 1



3, 3, 3, 2, 1, 0



Pf: $n=1$: simple graph can't have edges

$n>1$: Sufficiency: (If d' graphic, then d graphic)

$d: d_1 \geq \cdots \geq d_n \qquad \Delta = d_1$

Assume that

$$d': d_2-1, d_3-1, \ldots, d_{\Delta+1}-1, d_{\Delta+2}, \ldots, d_n$$

is the deg. sequence for a simple graph G'.

Add a new vertex adjacent to the vertices of degrees $d_2-1, d_3-1, \ldots, d_{\Delta+1}-1$; the resulting graph has deg. seq. d.

Necessity: Suppose d is the deg. seq. for a simple graph G.

Goal: create a simple graph G' w/ deg. seq. d'

Let $d_G(\underset{V(G)}{w}) = \Delta = d_1$

Let $S \subseteq V(G)$ w/ $w \notin S$ such that the vertices of S have degs. $d_2, \ldots, d_{\Delta+1}$

get rid of w & it's edges

If $N(w) = S$, $G' := G \setminus w$ is desired graph.

$N(w)$: neighborhood of w
set of all vertices adj. to w

Otherwise, $x \in S \setminus N(w)$

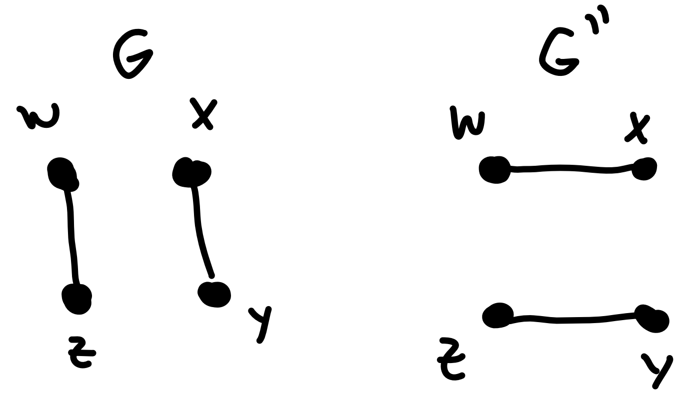
$z \in N(w) \setminus S$

Since $d(x) \geq d(z)$ and $w \in N(z) \setminus N(x)$,

there exists vertex $y \in N(x) \setminus N(z)$.

Let G'' be G with

- wz and xy deleted
- wx and yz added



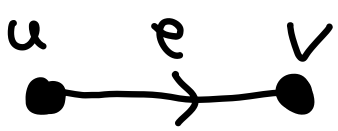
Then G'' has deg. seq. d, but a larger value of $|N(w) \cap S|$.

Repeating this argument, we eventually obtain a simple graph G^* w/ deg. seq. d. s.t. $N(w) = S$, and

G^* contains the desired subgraph G'. $\square$

§1.4: Directed Graphs

Def 1.4.2: A <u>directed graph</u> or <u>digraph</u> is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices



"e has endpoints u and v"

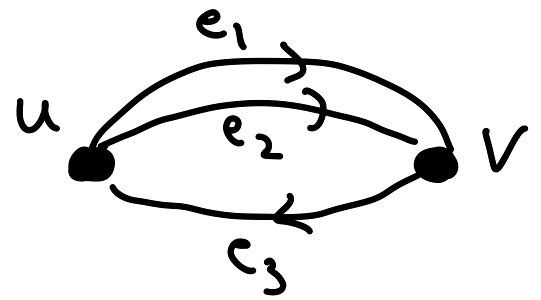
"e goes from u to v"

"e has tail u and head v"

"$u \to v$" or "$u \xrightarrow{e} v$"

Most basic def'ns are similar as for graphs.

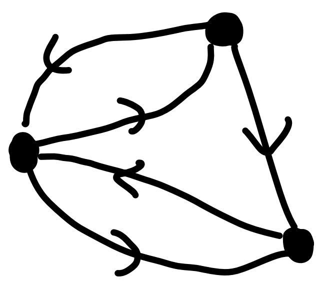
a) <u>Multiple edges</u> are edges w/ the same tail and head



e_1 & e_2 are multiple edges

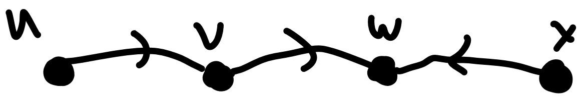
e_1 & e_3 are <u>NOT</u> mult. edges

b) A graph is simple if it has no $\underbrace{\text{loops}}_{\text{same as for graphs}}$ or multiple edges

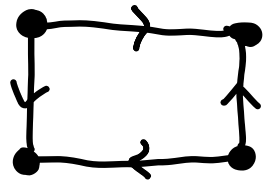


← simple

c) To be a path, cycle, walk, trail, circuit, you have to follow the edges tail to head

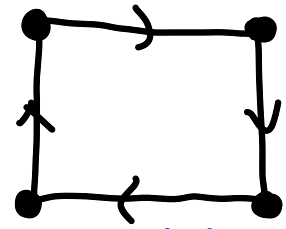


Not a path



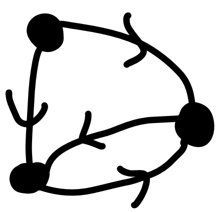
Not a cycle

Path!

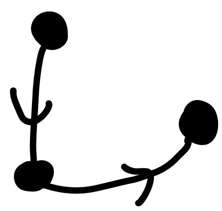


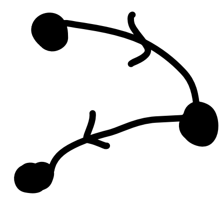
Cycle!

d) Subgraph, decomposition, union the same.

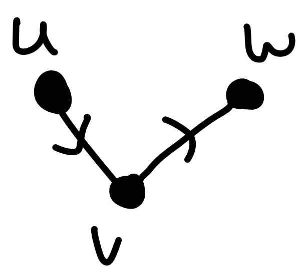


decomposes into

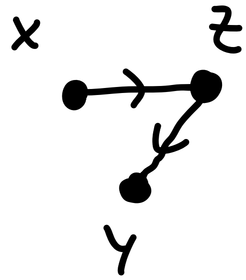




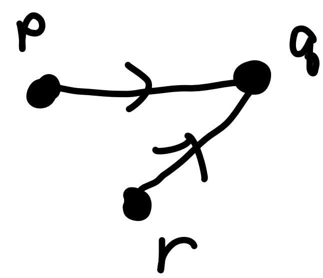
e) Isomorphism same, except edges have to point same direction



$\cong$



$\not\cong$

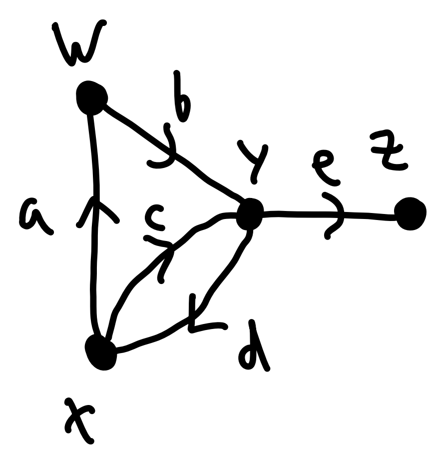


$f(u) = x$, $f(v) = y$, $f(w) = z$

f) The (i,j) entry of the adjacency matrix is the number of edges from v_i to v_j

The (i,j) entry of the incidence matrix of a loopless graph is $+1$ if v_i is the tail of e_j and -1 if v_i is the head of e_j

Class activity:



G

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} & 0 & & \\ 1 & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

$A(G)$

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} -1 & & & & \\ 1 & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

$M(G)$

g) For a vertex v,

$d^+(v)$: outdegree, # edges w/ tail v

$d^-(v)$: indegree, # edges w/ head v

$\delta^\pm(G)$: min out/indegree, $\Delta^\pm(G)$: max out/indegree

Successor: a vertex w s.t. $\exists$ an edge $v \to w$

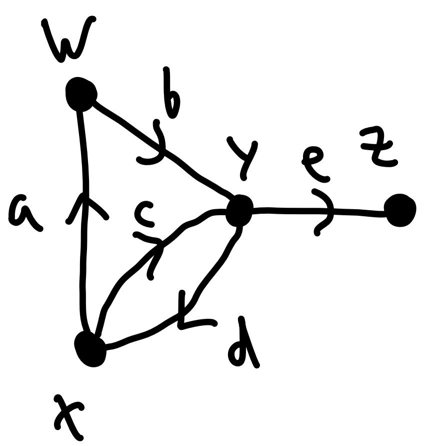
Predecessor: a vertex u s.t. $\exists$ an edge $u \to v$

$N^+(v)$: Out-nbhd/sucessor set, set of successors of v

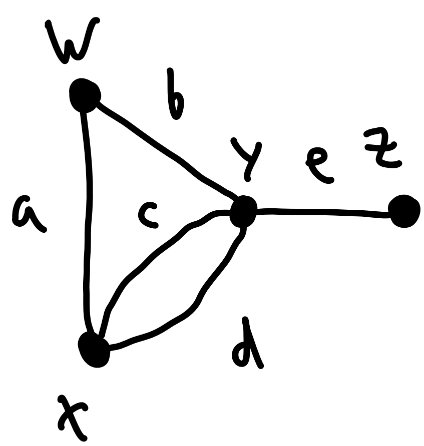
$N(v)$: In-nbhd/predecessor set, set of predecessors of v

Degree-sum formula: $e(G) = \sum_{v \in V(G)} d^+(v) = \sum_{v \in V(G)} d^-(v)$

h) The <u>underlying graph</u> of a digraph D is the graph G obtained by removing directions

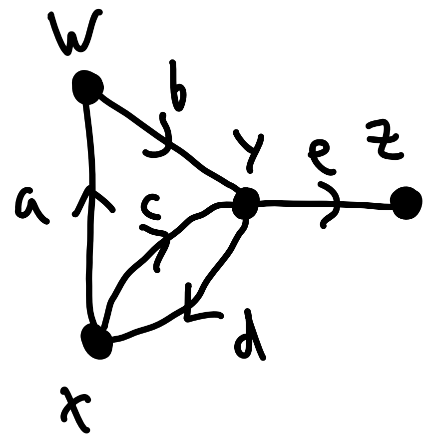


D



G

i) A digraph is weakly connected if the underlying graph is connected, and strongly connected if $\exists$ path from u to v $\forall$ vertices u, v



Thm 1.4.24: D: digraph

D has an Eulerian circuit $\iff$ a) $d^+(v) = d^-(v)$ $\forall v \in V(D)$

b) the underlying graph has ≤ 1 nontrivial component

D has an Eulerian trail $\iff$ a) $\sum_{v \in V(D)} |d^+(v) - d^-(v)| \leq 2$

b) the underlying graph has ≤ 1 nontrivial component