No new announcements today

Prop 1.3.15: If G is simple of order n, and $\delta(G) \geq \frac{n-1}{2}$, then G is connected

Pf: Let $u, v \in V(G)$. We will show that u and v are in the same conn. cmpt. of G. If u, v adjacent, done.

So assume they aren't adjacent. We'll show that they have a common neighbor.

Since G is simple,

$$|N(u)| = d(u) \geq \delta(G) = \frac{n-1}{2}$$

and similarly for $|N(v)|$.

By inclusion - exclusion,

$$\underbrace{|N(u) \cup N(v)|}_{\leq n-2} = \underbrace{|N(u)|}_{\geq \frac{n-1}{2}} + \underbrace{|N(v)|}_{\geq \frac{n-1}{2}} - |N(u) \cap N(v)|$$

since $u, v \notin N(u) \cup N(v)$

So $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)|$

$$\geq \frac{n-1}{2} + \frac{n-1}{2} - (n-2) = 1 \qquad \square$$

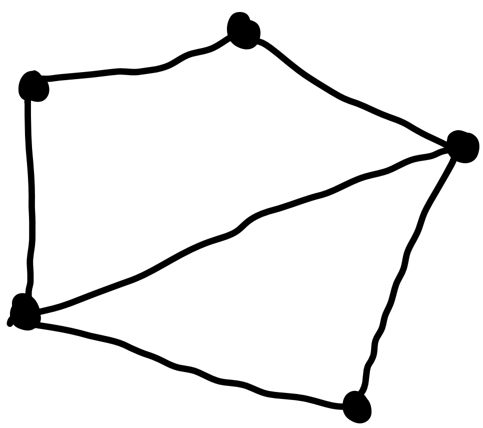
Recall: If $S \subseteq V(G)$, the induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set is

$$\{e \in E(G) \mid \text{both endpoints of } e \text{ are in } S\}$$

Def 1.3.22: G is H-free if G has no induced subgraph isomorphic to H.

Ex: By Konig's Theorem, bipartite graphs have no odd cycles. Therefore, if G is bipartite, G is C_{2k+1}-free for all k

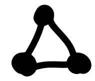
Note: being H-free is <u>not</u> the same as having no subgraph isomorphic to H.

E.g.:  is C_5-free

simple graph

except: G is K_j free $\Leftrightarrow$ G has no subgraph isom. to K_j

Mantel's Theorem [1907]: The maximu number of edges in an n-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$

$C_3 \cong K_3$ 

Pf: First, we exhibit a Δ-free graph w/ $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges: $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. This graph is bipartite, so Δ-free by Konig's Thm., and it has

$$\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \begin{cases} n^2/4, & n \text{ even} \\ \underbrace{\frac{n-1}{2} \frac{n+1}{2}}_{\frac{n^2-1}{4}}, & n \text{ odd} \end{cases} = \lfloor n^2/4 \rfloor \text{ edges}$$

For the converse, let G be an n-vertex Δ-free simple graph. Let $x \in V(G)$ w/ $d(x) = \underbrace{\Delta(G)}_{=:k}$

Since G is $\triangle$-free, $N(x)$ is an independent set, so every edge has ≥ 1 endpoint not in $N(x)$.

Therefore,

$$e(G) \leq \sum_{v \notin N(x)} d(v) \leq (n-k)k \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2}{4} \right\rfloor$$

e.g. $1 \cdot 6 \leq 2 \cdot 5 \leq 3 \cdot 4 = 4 \cdot 3$
$\geq 5 \cdot 2$
$\geq 6 \cdot 1$

Pf: $(k+1)(n-k-1) - k(n-k)$
$= n - 2k + 1$

$$\begin{cases} \geq 0, & \text{if } k < n/2 \\ < 0, & \text{if } k > n/2 \end{cases}$$

$\square$

Def: 1.3.27:

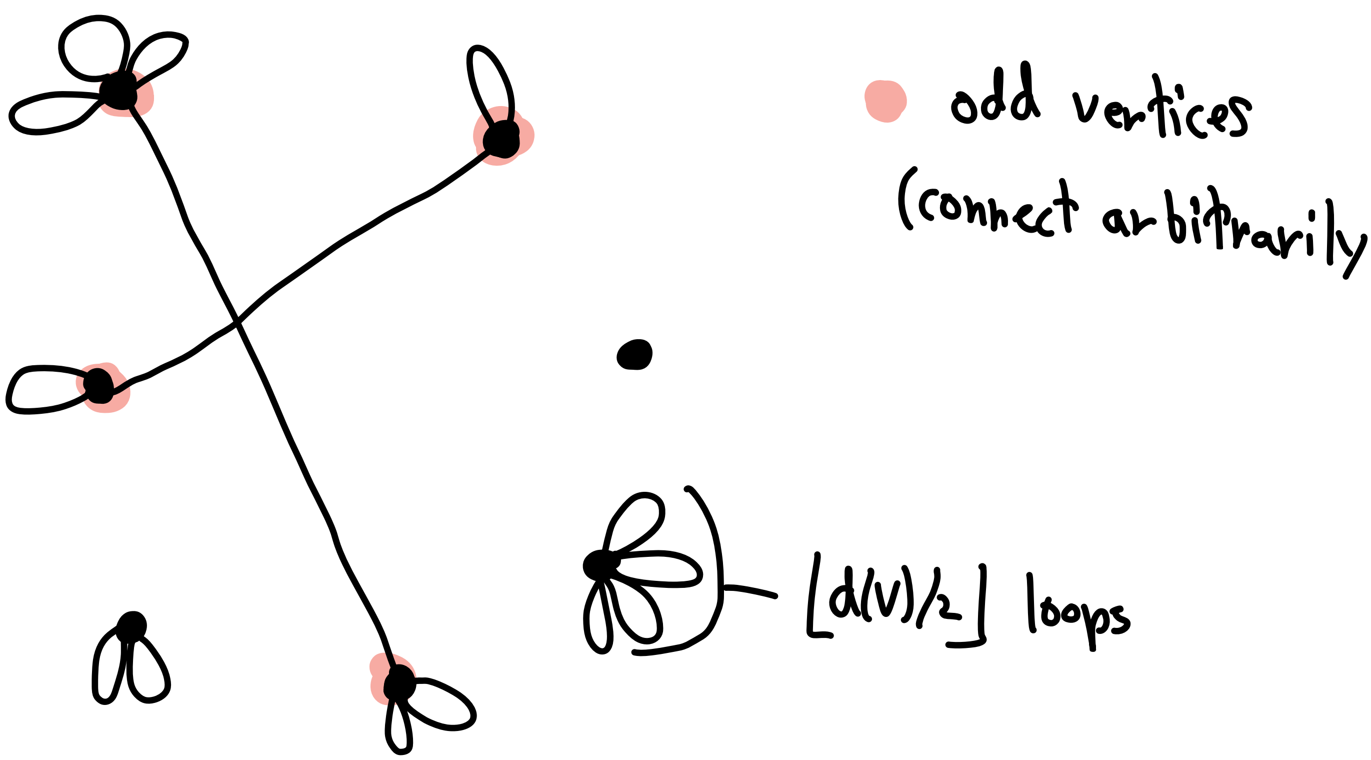
The <u>degree sequence</u> of a graph is a (usually weakly decreasing) list of the vertex degrees: $d_1, d_2, \ldots, d_n$

Question: Which sequences are the degree sequence

of some $\begin{cases} \text{a) graph? } \checkmark \\ \text{b) simple graph? "graphic"} \end{cases}$

Prop 1.3.28: A list $d_1, \ldots, d_n$ is the degree sequence of a graph iff $\sum d_i$ is even.

"Proof" by picture:

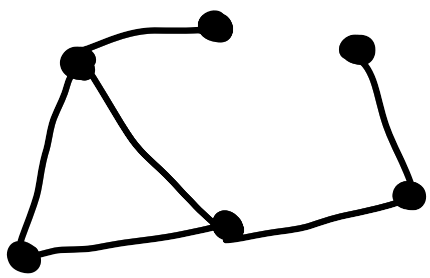


Havel-Hakimi Theorem:

a) For 1 vertex, the only graphic sequence is $d_1 = 0$

b) A list d of $n > 1$ integers is graphic iff d' is graphic, where d' is obtained by deleting the largest element Δ and subtracting 1 from its next Δ largest elements

Ex:



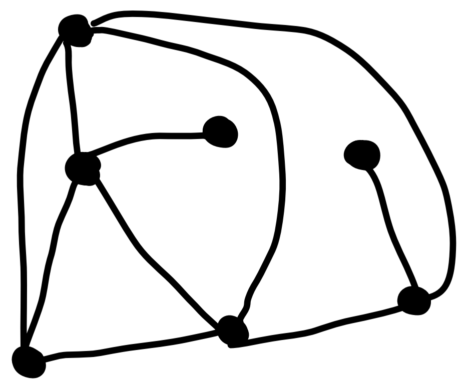
3,3,2,2,1,1 is graphic

So 4,4,4,3,3,1,1 is graphic

Since $\cancel{4}, 4, 4, 3, 3, 1, 1$

$\quad -1 \; -1 \; -1 \; -1$

$\quad 3, 3, 2, 2, 1, 1$



Pf: $n=1$: Simple graph can't have edges

$n>1$: (Next time)

Sufficiency: (If d' graphic, then d graphic)

(Use algorithm from example)

Necessity: (If d graphic, then d' graphic)