Announcements:

- HW2 posted (due Wed. 9am)
- No class Monday!

Recall: Konig's Theorem [1936]: G: graph

G is bipartite $\iff$ G has no odd cycle

Proved $\Rightarrow$

When G is connected, reduced $\Leftarrow$

to the following claim:

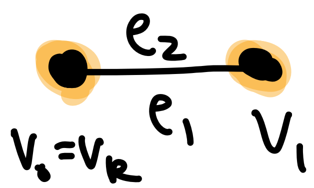
Claim: Every closed odd walk contains an odd cycle.

Pf of Claim: Induction on the length ℓ of a closed odd walk W:

Base case: $\ell = 1$. Must be a loop i.e. a 1-cycle

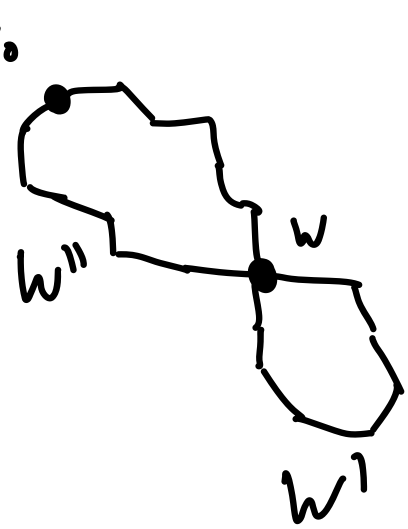
Inductive step: Suppose $\ell > 1$, and assume the claim holds for all shorter closed odd walks.

If W has no repeated vertex (other than $v_0 = v_k$), then W is an odd cycle.



If w is a repeated vertex, then W is of the form

$$v_0, \dots, \overset{W''\ \curvearrowright}{\underbrace{w, e_i, \dots, w}_{W'}}, e_j, \dots, v_0$$



The walks

$$W' = w, e_i, \dots, w$$

and

$$W'' = v_0, \dots, w, e_j, \dots, v_0$$

are strictly shorter closed walks contained in W. Since W is odd, one of W' and W''

is odd, and by the inductive hypothesis contains an odd cycle, which is therefore also contained in W

Return to proof of theorem:

$\Leftarrow$) (cont.)

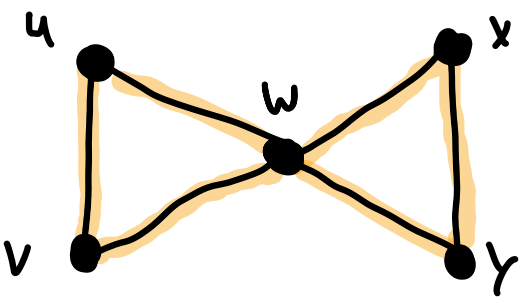
If G is any graph w/ no odd cycle, we have just shown that each of its conn. components is bipartite. Therefore, G is bipartite, with each part being the union of one part of each component of G $\square$

Eulerian circuits

Def 1.2.24:

a) A circuit is a closed trail. Two circuits are equivalent if they're the same up to cyclic order and reversal (book slightly different)

Class activity: equivalent or not?



a) u, v, w, x, y, w, u

b) w, y, x, w, v, u, w

c) v, w, x, y, w, u, v

d) u, v, w, y, x, w, u

e) w, u, v, w, x, y, w

b) An Eulerian $\left\{\begin{array}{l}\text{trail}\\ \text{circuit}\end{array}\right.$ is a $\left\{\begin{array}{l}\text{trail}\\ \text{circuit}\end{array}\right.$ containing all the edges

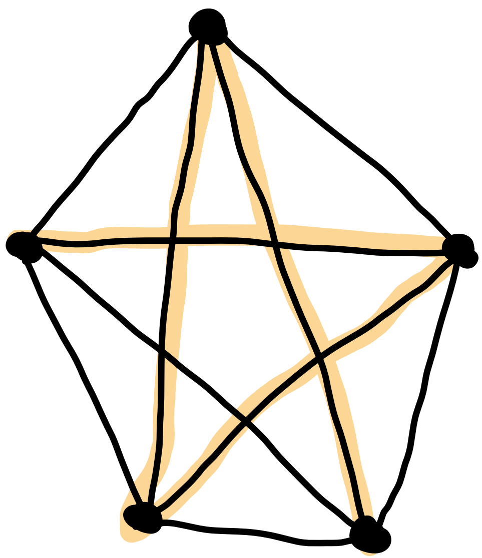
c) A graph is $\left\{\begin{array}{l}\text{even}\\ \text{odd}\end{array}\right.$ if all vertex degrees are $\left\{\begin{array}{l}\text{even}\\ \text{odd}\end{array}\right.$

(Note: loops count double for degree)

d) A maximal path is a path not contained in a longer path in G

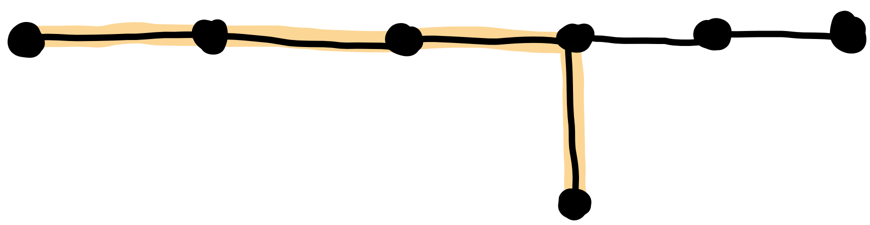
Class activity: Which of these are maximal paths?

a)



✓

b)



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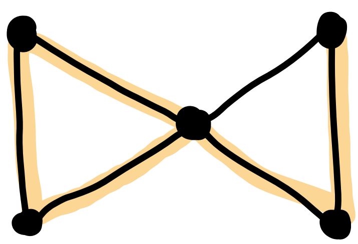
c)

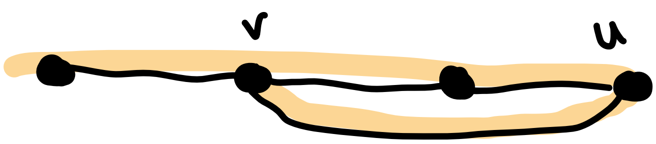


✗ not a path

Lemma 1.2.25: If $\deg v \geq 2$ for all $v \in V(G)$, then G contains a cycle.

Pf: Let $P \subseteq G$ be a max'l path w/ endpoint u. Every neighbor of u is in P since otherwise P wouldn't be maximal.

Since $\deg u \geq 2$, u has a neighbor v and an edge e from u to v such that $e \notin P$, but $v \in P$. Therefore, $P \cup e$ contains a cycle by taking $e \cup$ the shortest path in P from u to v. $\square$



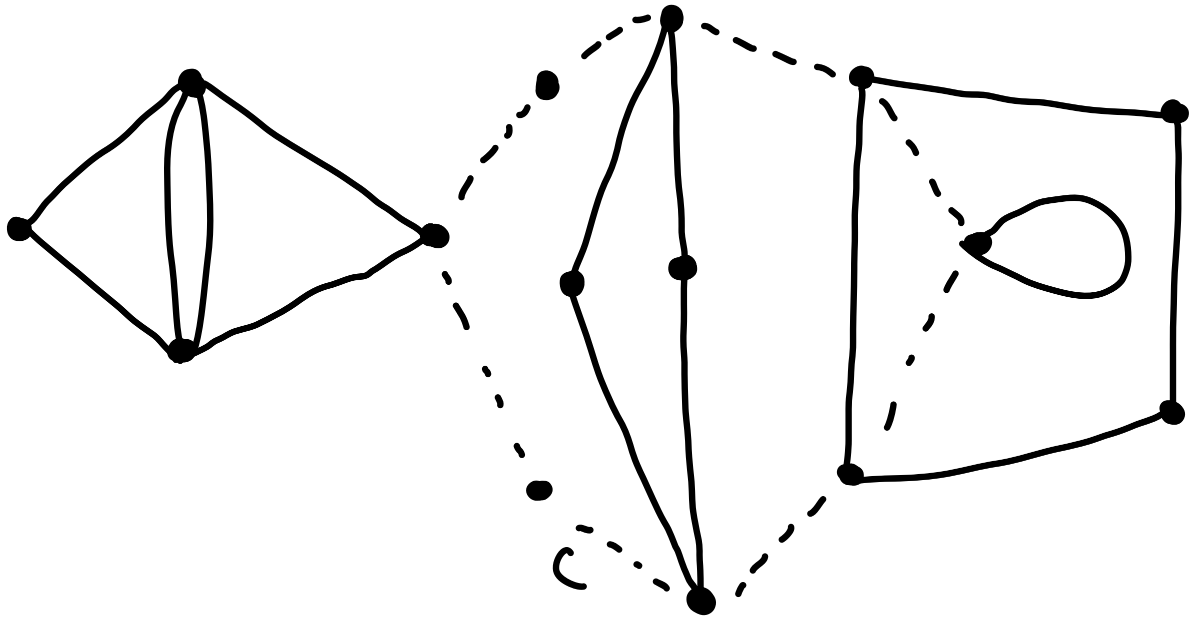
Thm 1.2.26 [Euler]:

G has an Eulerian circuit $\iff$ a) G has ≤ 1 "nontrivial" connected component AND b) G is even

containing edges

Pf: $\Rightarrow$) Every circuit C of G uses an even number of edges incident to any vertex v, since each passage of C thru v uses two incident edges, and the first edge is paired w/ the last at the first vertex.

If C is an Eulerian circuit, $E(C) = E(G)$, so every vertex of G has even degree. Furthermore, edges can be in the same walk only when they lie in the same component, so G must have at most one nontrivial component.

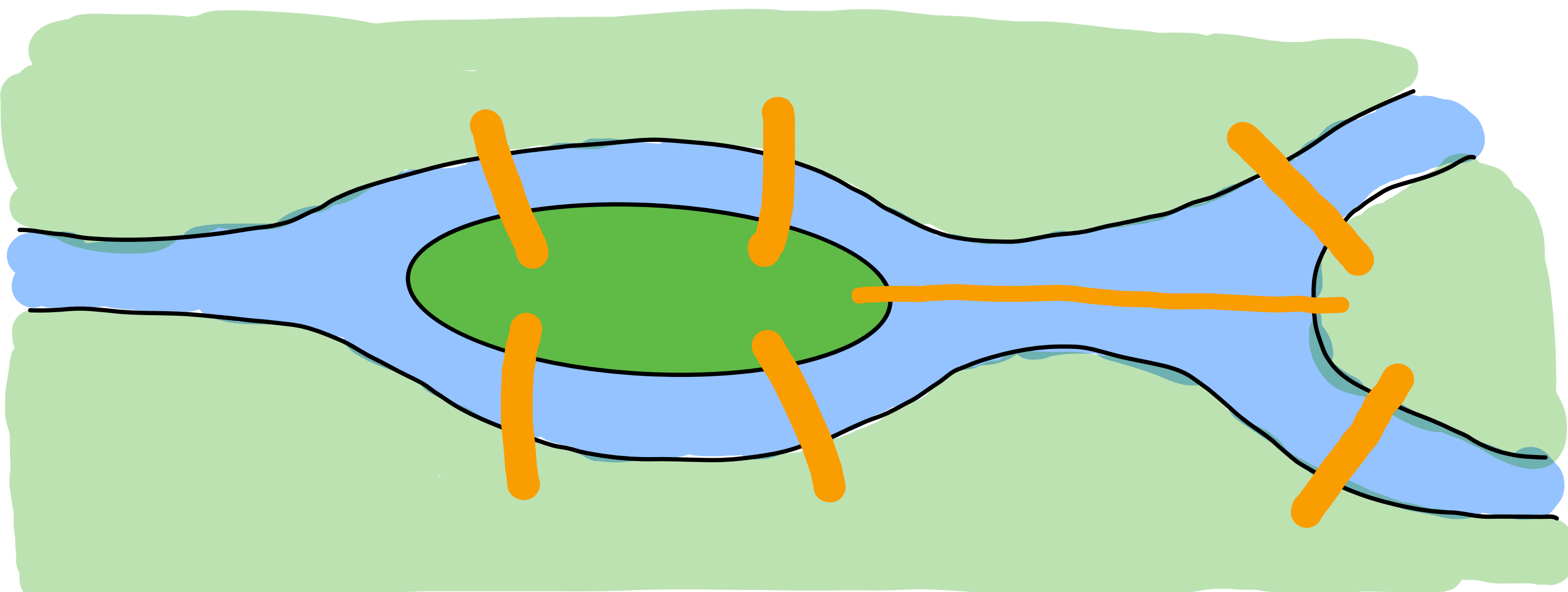


Def 1.1.32: A decomposition of G is a list of subgraphs s.t. each edge appears in exactly one subgraph from the list

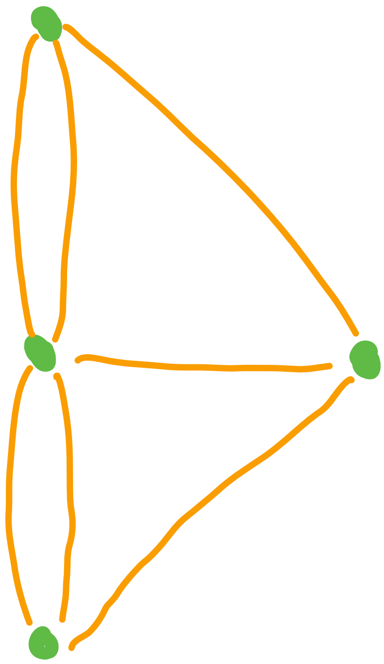
Corollary (Prop 1.2.27): Every even graph decomposes into cycles.

Pf: In the previous proof, G decomposes into G' and C; use induction on $|E(G)|$. $\square$

Bridges of Königsberg (redux)



Question: Can we cross each bridge exactly once?



Answer: No, since the corresponding graph is not even (in fact, it's odd).

Cor:

G has an Eulerian ~~circuit~~ trail $\Longleftrightarrow$ a) G has ≤ 1 "nontrivial" connected component

AND

b) ~~G is even~~ G has at most two $\underbrace{\text{odd vertices}}_{\text{vertices of odd degree}}$

Pf: $\Rightarrow$) If the trail is closed, it's a circuit. Otherwise, the starting and ending vertices have odd degree; add an edge between them and apply Thm. 1.2.26.

$\Leftarrow$) If G has no odd vertices, by Thm. 1.2.26 it has an Euler circuit. Otherwise, add an edge between the two odd vertices, and the resulting graph has an Euler circuit (again, by Thm. 1.2.26). Remove the edge you just added, and it becomes an Euler trail. $\square$

Cor: The Königsberg bridge graph doesn't have an Euler trail. $\square$