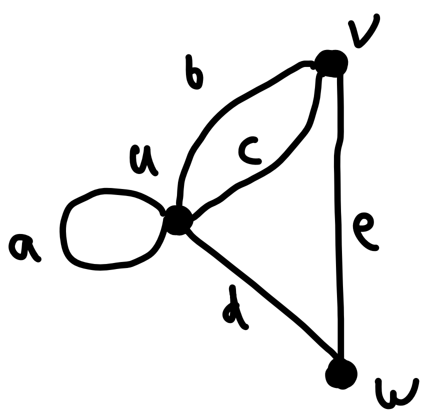
Announcement: H/w 2 will be posted later today

Today: • Connectivity, cut-edges, and cycles
• Konig's Theorem

Recall: Lemma 1.2.5: Every u,v-walk contains a u,v-path

Key step: If w appears more than once, delete everything btwn first and last occurence (see notes from last time for full proof)

Ex:



u, ~~a, u, c, v, b, u~~, d, w

u, d, w

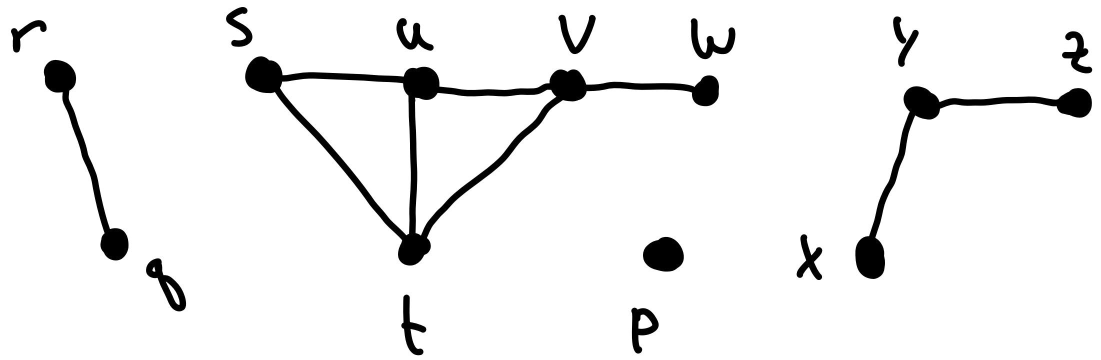
Def 1.2.6 / 1.2.8:

a) G is <u>connected</u> is $\forall u, v \in V(G)$, G contains a u,v-path (or walk or trail)

b) The <u>(connected) components</u> of G are its maximal connected subgraphs

c) An <u>isolated vertex</u> is a vertex of deg 0

Ex 1.2.9:

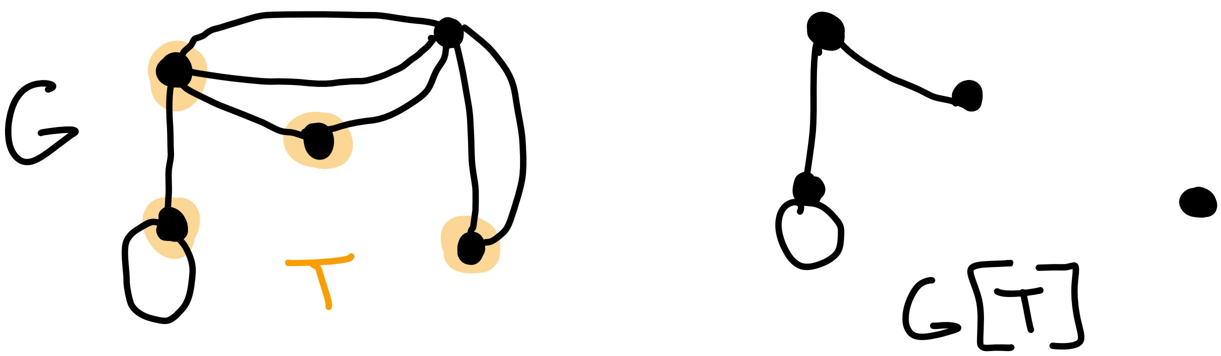


Remark 1.2.7: "u and v are in the same connected component" is an equivalence rel'n

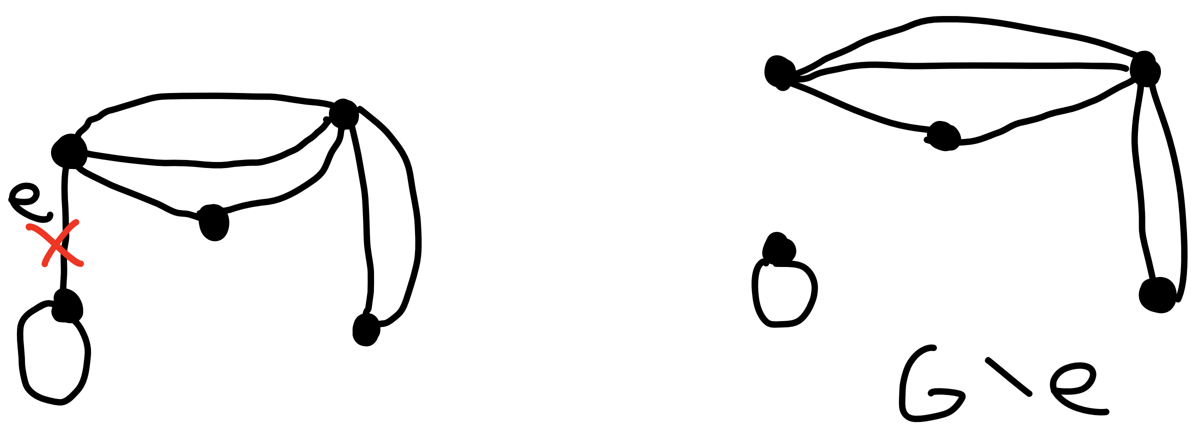
Def 1.2.12:

a) If $T \subseteq V(G)$, the *induced subgraph* $G[T]$ is the graph w/ vertex set T and edge set

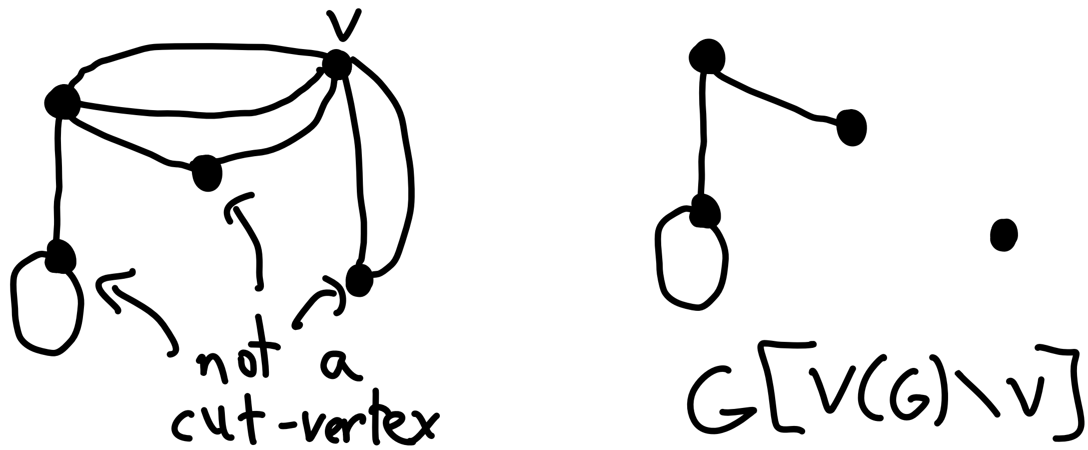
$$E(G) \cap \{\text{edges w/ both endpoints in } T\}$$



b) An edge $e \in E(G)$ is a *cut-edge* if the graph $G \setminus e := (\underbrace{V(G)}_{\text{vertex set}}, \underbrace{E(G) \setminus e}_{\text{edge set}})$ has one more conn. cmpt. than G



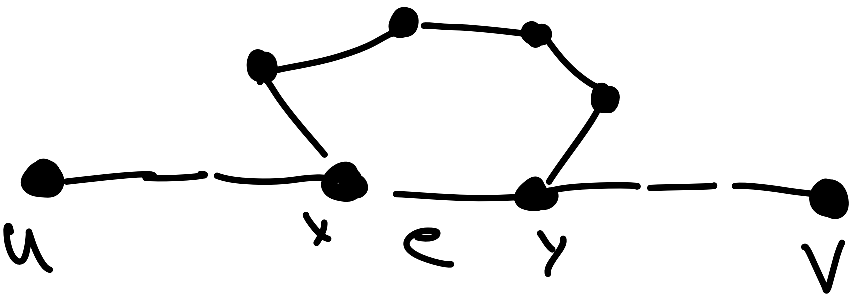
c) A vertex $v \in V(G)$ is a cut-vertex if $G[V(G) \setminus v]$ has more conn. cmpts. than G



Thm 1.2.14: An edge $e \in E(G)$ is a cut-edge iff it belongs to no cycle

Pf: Let e have endpoints x and y. First, assume G is connected.

$\Longrightarrow$) If E is a cut-edge, choose $u, v \in V(G)$ s.t. u & v are in separate conn. components of $G \setminus e$. Therefore, every u,v-path P contains e, so in particular, P contains x and y. If there is a cycle $C \subseteq G$ containing e, then $C \setminus e$ is an x,y-path, and



replacing e in P with $C \setminus e$ gives a u,v walk in $G \setminus e$, which contradicts the assumption that u,v are in separate components.

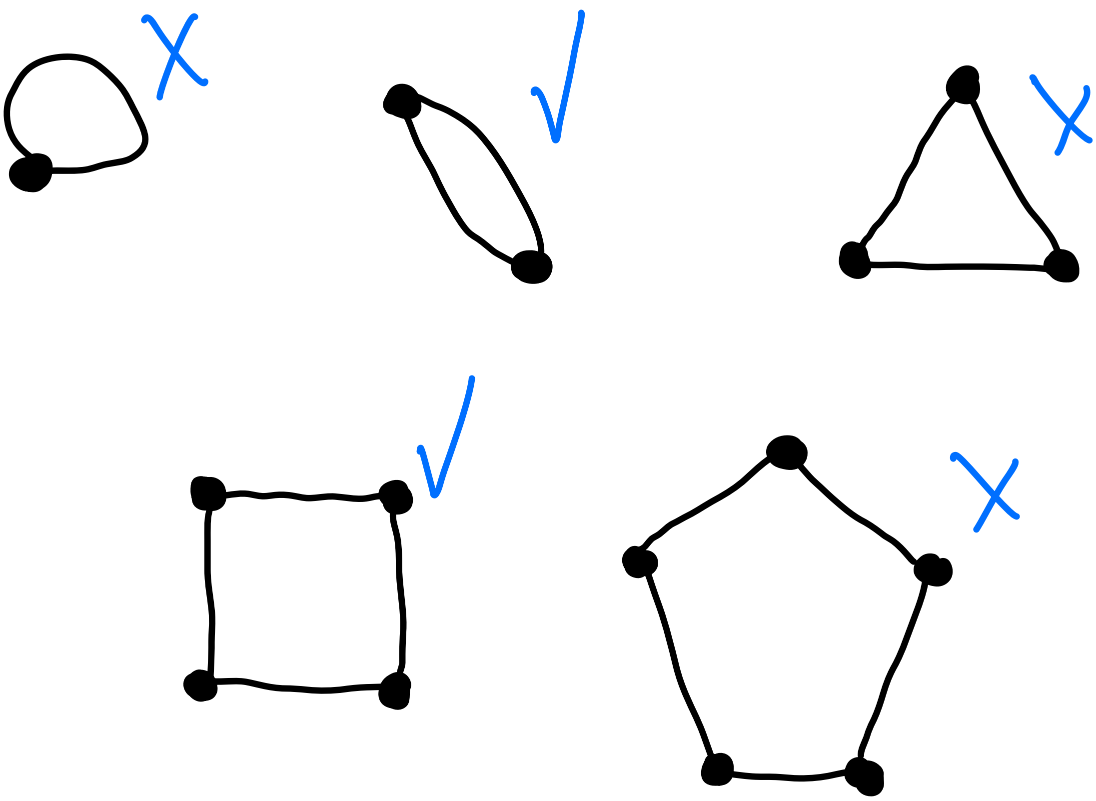
$\Longleftarrow$) If e belongs to no cycle, there is no x,y-path in $G \setminus e$, so $G \setminus e$ is disconnected, and so e is a cut edge.

If G is disconnected, apply the argument above to the conn. component of G containing e. $\square$

Next goal: Characterize bipartite graphs using cycles

Class activity (toy example):

Which cycles C_n are bipartite?



Proposition [Us, 2023]: C_n is bipartite if and only if n is even.

Konig's Theorem [1936]: G: graph

G is bipartite $\iff$ G has no odd cycle

Pf: $\Rightarrow$) Suppose G is bipartite, and write $V(G) = S \sqcup T$, where

disjoint union

S and T are independent sets.

Consider a walk

$$W = v_0, e_0, v_1, \ldots, e_k, v_k$$

in G. B/c G is bipartite, $v_0, v_1, \ldots, v_k$ alternate btwn. elements of S and T.

Therefore, if k is odd v_0 and v_k are in opposite sets, so all closed walks are even length. But cycles are closed walks, so all cycles in G are even length.

$\Leftarrow$) Suppose G has no odd cycle.

First, we consider the case where G is connected. Choose $u \in V(G)$ and let

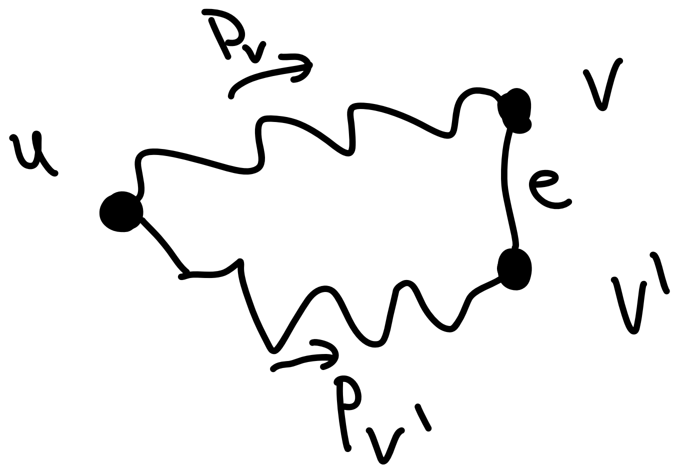
$$S = \{v \in V(G) \mid \text{the min'l length of a } u,v\text{-walk is even}\}$$

$$T = \{v \in V(G) \mid \text{the min'l length of a } u,v\text{-walk is odd}\}$$

Since G is connected, $S \sqcup T = V(G)$

For all $v \in V(G)$, let P_v be a minimal length walk from u to v.

If v, v' are either both in S or both in T, then if e is an edge w/ endpoints v and v', the walk given by P_v followed by e followed by the reverse of $P_{v'}$ is a closed odd walk



Since we know G has no odd cycles, we will know that v, v' are not adjacent, and therefore that G is bipartite, once we prove the following

Claim: Every closed odd walk contains an odd cycle.