Announcements:

- Today: exam review
- Review session: Tues 12/12 5:00 – close, 156 Henry Admin. Bldg.
- Final exam: Thurs 12/14, 8:00 – 11:00 am, 132 Berier Hall

  TWO reference sheets (2x front and back) allowed

  Cumulative: everything from the course is fair game

  See Monday's email for full policies

Partial list of topics:

Basic def'ns (e.g. vertex, edge, simple graph, etc.)

Basic examples (e.g. K_n, C_n, P_n, $K_{r,s}$, small examples)

Classes (e.g. trees, bipartite graphs, weighted graphs, digraphs)

Paths/cycles/walks/trails/circuits

Ch. 1 Theorems:

Eulerian circuits/trails for graphs/digraphs

Mantel's Theorem (max. edges in $\triangle$-free graph)

Konig's Theorem (bipartite $\leftrightarrow$ no odd cycles)

Havel-Hakimi Theorem

Trees:

- Equiv. def'ns
- Prüfer code & Cayley's formula
- Spanning subgraphs & spanning trees
- Matrix tree thm.
- Kirchoff's Laws and Kirchoff's Thm.

Algorithms:

- Kruskal (min. wt. spanning tree)
- Dijkstra (distances)
- Gale-Shapley (stable matching)
- Algorithmic thinking

Matchings:

- Perfect vs. $\overbrace{\text{maximum vs. maximal}}^{\text{general concept}}$
- M-alt. paths & M-aug. paths
- Theorems: Berge, Hall, Tutte, Berge-Tutte, Petersen x2
- Relationships btwn. matchings, vertex/edge covers, and indep. sets
- k-factors

Vertex/edge connectivity:

- Def'ns
- Whitney's Thm.
- Different characterizations of 2-connectivity and 2-edge-connectivity
- Digraph vertex/edge connectivity
- Menger's Theorem (4 versions)

Max-flow, min-cut theorem

- Def'ns
- Theorem itself
- Ford-Fulkerson algorithm

Connections between: flows, cuts, (edge)-disjoint paths, matchings, indep. sets, vertex/edge covers, etc.

Vertex coloring

- Def'ns (e.g. Chromatic number, k-criticality)
- "Easy bounds", and more difficult ones (e.g. Brooks' Thm.)
- Greedy coloring: Algorithm & Consequences

- Mycielski's construction and theorem
- Chromatic polynomial
  - Values/how to compute for small graphs
  - Deletion-contraction recurrence

Planar graphs

- Planar graph vs. plane graph vs. planar embedding
- Dual graph & vertices/edges/faces (degree sum ×2)
- Euler's formula & consequences
  - Polyhedra
  - $e(G) \leq 3n(G) - 6$
  - Nonplanarity of K_5 & $K_{3,3}$
- Triangulations (equiv. defns)
- Kuratowski's thm. and proof of easy direction
- k-color theorems and proof technique

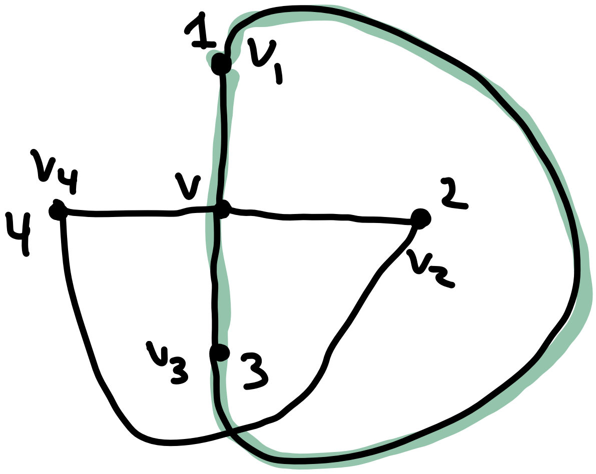
Examples:

1) Let G be a planar graph w/ ≤ 11 vertices. Without using the 4-color theorem, prove that G is 4-colorable.

Pf: Suppose for a contradiction that G is a planar graph w/ ≤ 11 vertices s.t. G is not 4-colorable, but every planar graph w/ $< n$ vertices is 4-colorable. Clearly, $n \geq 5$. By Thm. 6.1.23, $e(G) \leq 3n(G) - 6$, so by the deg. sum formula,

$$\sum_{v \in V(G)} d(v) = 2e(G) \leq 6n(G) - 12 < 5n(G)$$

since $n < 12$. Therefore, G has a vertex v of degree ≤ 4. By assumption, $G \setminus v$ is 4-colorable. Choose a 4-coloring of $G \setminus v$. Since G is not 4-colorable, $N(v)$ must have (4) distinct colors. WLOG, assume the neighbors of v are colored 1,2,3,4 clockwise from top.



Let G_{ij} be the induced subgraph of G by vertices of colors i and j. Let P_{ij} be a path in G_{ij} from v_i to v_j (if one exists).

If P_{13} doesn't exist, swapping 1 and 3 in the component of G_{13} containing v_1, and coloring v color 1 yields a proper 4-coloring of G; hence P_{13} must exist. Similarly, so must P_{24}.

However, if C is the cycle formed by traversing P_{13} followed by v followed by v_1, then v_2 is inside C while v_4 is outside C, or vice-versa. By the Jordan curve thm., P_{24} must intersect C, but this is impossible since G is planar and G_{13} and G_{24} are disjoint, a contradiction. $\square$

2) Use network flows to prove that for any two nonadjacent vertices $x, y \in V(G)$, G: graph

$$\underbrace{\kappa(x,y)}_{\substack{\text{size of minimum} \\ x,y\text{-cut}}} = \underbrace{\lambda(x,y)}_{\substack{\text{max. number of} \\ \text{internally-disjoint } x,y\text{-paths}}}.$$

Integrality Theorem: If a network has integer capacities, $\exists$ a maximum feasible flow where every edge flow is an integer.

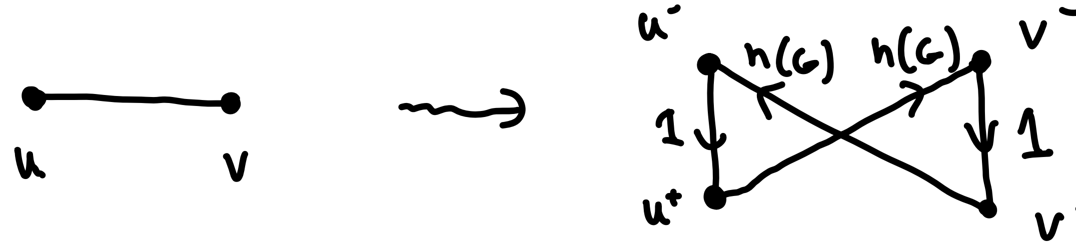
Pf: If $\exists k$ internally-disjoint x,y-paths, then every x,y-(vertex) cut must contain ≥ 1 vertex from every path, so has $\geq k$. Thus, $\kappa(x,y) \geq \lambda(x,y)$.

Conversely, let D be the following network, with source x and y. Starting w/ G,

- replace each vertex $w \in V(G) \setminus \{x,y\}$ w/ a pair of vertices w^-, w^+ with an edge of capacity 1 from w^- to w^+.

- replace each $uv \in E(G)$ w/ the edges u^+v^- and v^+u^-, each w/ capacity $n(G)$.

e.g.



By the max-flow, min-cut thm., the value k of a maximum feasible flow f in D equals the capacity a minimum edge cut $[S,T]$ in D.

Rest will be posted in notes

Idea: show that min. edge cut gives a same-size vertex x,y-cut in G

show that max. flow gives a set of same-size int.-disjoint x,y-paths in G.

Rest of proof:

Since all capacities are integers, we can assume all edge flows in f are integers ("Integrality theorem", Cor. 4.3.12).

Since v^- has out-capacity 1, at most one edge into v^- has nonzero flow. Hence, by following the flows, we obtain k internally-disjoint paths in D, and thus k internally-disjoint paths in G.

On the other hand, any minimum edge cut is a subset F of $\{w^-w^+ \mid w \in V(G)\}$ since this is an edge cut of capacity $n(G)-2$ and every other edge has capacity $n(G)$.

Deleting w^-w^+ in D corresponds to deleting w in G, and F is an edge cut if and only if the corresponding vertices are an x,y-cut in G.

Therefore, $\kappa(x,y) \leq k \leq \lambda(x,y)$, so they are equal.