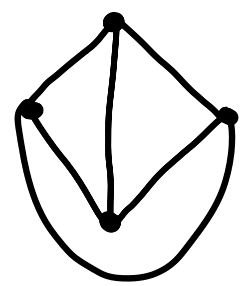
Announcements:

- Quiz 4: Monday in class (covers Ch. 6)
- Exam review: Wed., plus some more review later
- Final exam: Thurs 12/14, 8:00-11:00am, 132 Berier Hall (cumulative!)

Recall: "k-color theorem" means "every planar graph is k-colorable."

No 3-color theorem. Counterexample: K_4



Last time: 6-color theorem ✓

Five-color theorem [Heawood, 1890]: Every planar graph is 5-colorable.

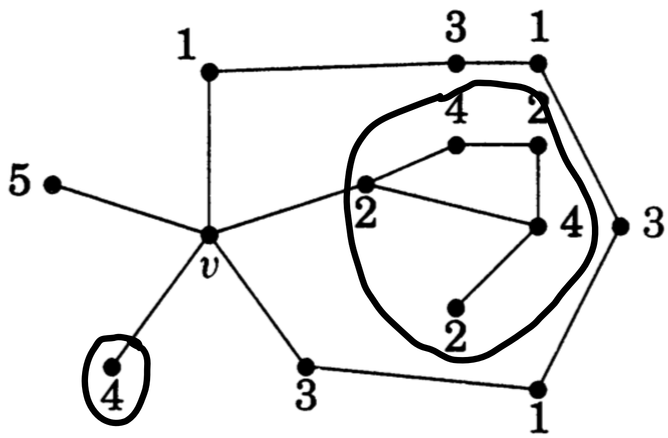
Pf: Induction on $n(G)$.

Base case: $n(G) \leq 5$. Can color every vertex a diff. color.

Inductive step: $n(G) > 5$. Let $v \in G$ have degree ≤ 5 (see pf. of 6-color thm). By the inductive hyp.,

$G \setminus v$ is 5-colorable, and if $d(v) \leq 4$, G is 5-colorable (see pf. of 6-color thm.). So assume $d(v)=5$, and let f be a proper 5-coloring of $G \setminus v$.

Assume that G is not 5-colorable. Then, let $v_1, \ldots, v_5$ be the neighbors of v in clockwise order. Permute the colors so that $f(v_i)=i$ for all $i=1,\ldots,5$.



For all $i,j \in \{1,\ldots,5\}$, let G_{ij} be the subgraph of $G \setminus v$ induced by vertices of colors i and j. Swapping the colors $i \leftrightarrow j$ on any conn. component of G_{ij} yields another proper coloring of $G \setminus v$. If v_i and v_j are in diff. components of G_{ij}, then we can do this swap on the comp. of G_{ij} containing v_i, leaving a proper 5-coloring of $G \setminus v$ where v has no neighbor of color i (and two of color j); coloring v color i yields a proper 5-coloring of G. Since we are assuming G is not 5-colorable, this means that for all i, j $\exists$ path P_{ij} from v_i to v_j in G_{ij}.

Let C be the cycle formed by adding v to P_{13}. v_2 is inside C, while v_4 is outside C or vice-versa, so by the Jordan curve theorem, $P_{2,4}$ must cross C. Since G is planar, this means that P_{24} and C share a vertex; however every vertex of P_{24} has color 2 or 4 and every vertex of C has color 1 or 3 (or is v), so this is impossible, a contradiction. $\square$

Let's take these ideas to the 4-color problem

Def 6.3.2:

a) A <u>configuration</u> in a planar triangulation is a cycle C called the <u>ring</u> together with the portion of the graph inside C.

b) For the 4-color problem,

i) a set of configurations is <u>unavoidable</u> if a minimal couterexample must contain a member of it.

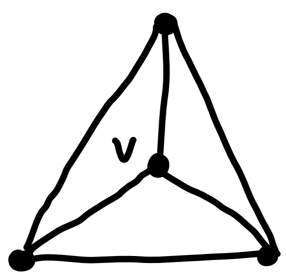
ii) a configuration is <u>reducible</u> if a planar graph containing it cannot be a min'l counterexample

Proof idea:

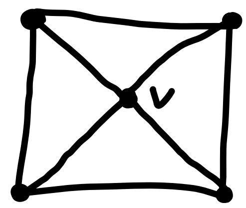
- Work w/ triangulations; for an arbitrary graph, simply remove some edges
- Find an unavoidable set of configurations
- Prove that each of these configurations is reducible

Four-Color Theorem: Every planar graph is 4-colorable

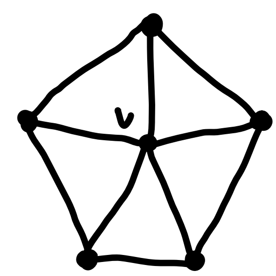
Pf [Kempe, 1879]: In a planar triangulation, $3 \le \delta(G) \le 5$, so the following set of configs. is unavoidable:



$\bullet 3$



$\bullet 4$

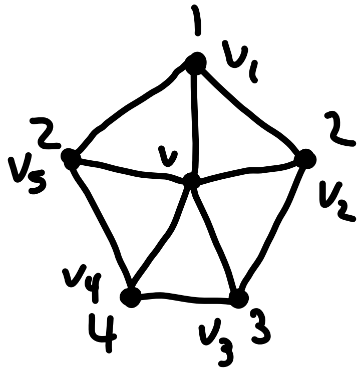


$\bullet 5$

Let G be a minimal counterexample, so that $G \setminus v$ is 4-colorable. If $d(v)=3$, color v any color diff. from its neighbors, so $\bullet 3$ is reducible.

If $d(v)=4$, same argument works as for the 5-color theorem, so $\bullet 4$ is reducible.

If $d(v)=5$, WLOG we can color $N(v)$ like this:

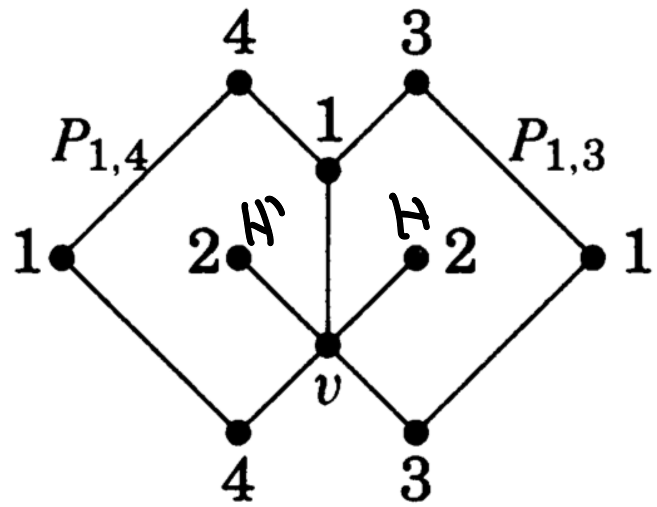


Again, let

G_{ij} = induced subgraph of $G \setminus v$ consisting of vertices of color i or j

P_{ij} = path from v_i to v_j in G_{ij} (if it exists/make sense)

P_{13} and P_{14} must exist or we can eliminate color 1 from $N(v)$ by swapping 1 and 3 (resp. 1 and 4) in the component of G_{13} (resp. G_{14}) containing v_1.



let H = component of G_{24} containing v_2

H' = component of G_{23} containing v_5

Notice that $v_4 \notin H$ and $v_3 \notin H'$.

So swap colors 2 & 4 on H

and swap colors 2 & 3 on H'

Now $N(v)$ has no vertex of color 2.

So let v be color 2 $\rightarrow$ proper 4-coloring of G.