Announcements:

- HW9 due tomorrow (Thurs. 10/30) at 9am (office hour today)
- Exam 3 graded

Mean: 62.9, Median: 64.5 (out of 95)

Std. dev.: 15.0

Problem Scores:

Q_1: 82% Q_3: 50%

Q_2: 50% Q_4: 75%

- Plan for rest of semester (rough!)

Wed 11/29: §6.1, §6.3 if time

Fri 12/1: §6.3

Mon 12/4: §6.3 (cont.) and Quiz 4

Wed 12/6: Final exam review

(Some sort of review session + office hours)

Thurs 12/14, 8:00-11:00am: Final exam!

132 Berier Hall (<u>not</u> one of our usual rooms!)

Recall:

Euler's Formula: Let G be a connected plane graph w/ n vertices, e edges, and f faces. Then,

$$n - e + f = 2$$

Last time: used this to study regular polyhedra

Today: a bunch of corollaries

Remark 6.1.22:

a) Since n and e don't depend on the planar embedding, neither does f.

b) Recall that the dual graphs of two different planar embeddings of G can be nonisomorphic. However, if the dual graph has n^* vertices, e^* edges, and f^* faces, $n^* = f$, $e^* = e$, $f^* = n$, so these numbers are independent of planar embedding.

c) For a graph w/ k conn. components, we have

$$n - e + f = k + 1$$

Thm 6.1.23:

a) If G is a simple planar graph w/ ≥ 3 vertices, then

$$e(G) \leq 3n(G) - 6$$

b) If G is also $\triangle$-free, then

$$e(G) \leq 2n(G) - 4$$

Pf: Assume G is connected; if it's not, add edges to form a conn. planar graph, and a), b) for G will follow from a), b) for that graph.

a) Since G is simple, every face has length ≥ 3 since faces (except poss. the ∞-face) are bounded by cycles when $n(G) \geq 3$.

By the deg.-sum formula for G^*,

$$2e(G) = \sum_{F:\text{face}} \ell(F) \geq 3\underbrace{f(G)}_{\substack{\#\text{ faces} \\ \text{of } G}}$$

By Euler's formula, this becomes:

$$2e(G) \geq -3n(G) + 3e(G) + 6,$$

and rearranging gives $e(G) \leq 3n(G) - 6$.

b) Do the exact same thing, but now face lengths are ≥ 4.

$$2e(G) = \sum_{F:\text{face}} \ell(F) \geq 4f(G)$$

$$2e(G) \geq -4n(G) + 4e(G) + 8$$

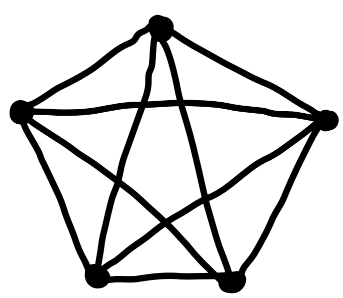
$$e(G) \leq 2n(G) - 4. \qquad \square$$

Corollary (6.1.24): K_5 and $K_{3,3}$ are nonplanar

(already proved this)

Pf: Class activity!

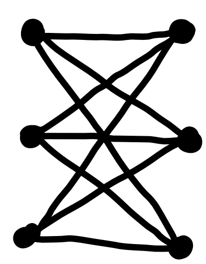
$$\left[\begin{array}{l} e(G) \leq 3n(G) - 6 \\ \triangle\text{-free: } e(G) \leq 2n(G) - 4 \end{array}\right]$$



$n = 5$

$e = 10$

$10 \leq 3(5) - 6 = 9$



$n = 6$

$e = 9$

$9 \leq 2(6) - 4 = 8$

Def 6.1.25:

a) A maximal planar graph is a simple planar graph that is not a spanning subgraph of another planar graph (i.e. adding edges makes the graph nonplanar)

b) A triangulation is a simple plane graph where every face boundary is a 3 cycle.

Prop 6.1.26: Let G be a simple n-vertex plane graph. The following are equivalent:

A) $e(G) = 3n - 6$

B) G is a triangulation

C) G is (an embedding of) a maximal planar graph.

Pf: A $\Leftrightarrow$ B: From pf of Thm. 6.1.23,

$$e(G) = 3n - 6 \Leftrightarrow 2e(G) = 3f(G)$$

$\Leftrightarrow$ Every face bdy has length 3

$\Leftrightarrow$ Every face bdy is a Δ

$\Leftrightarrow$ G is a triangulation.

$B \Leftrightarrow C$:

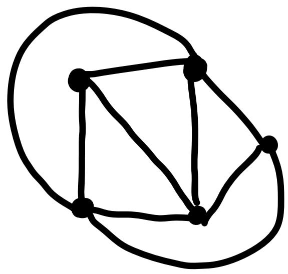
G is not a triangulation

$\Leftrightarrow$ G has a face bdy. longer than a Δ

$\Leftrightarrow$ $\exists$ nonadj. vertices bounding the same face

$\Leftrightarrow$ can add an edge and maintain planarity

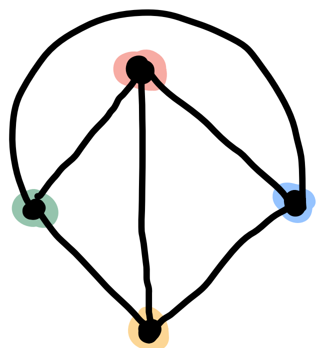
$\Leftrightarrow$ G is not maximal planar. $\blacksquare$



Recall our main question for this section: what is the maximum number of colors needed to give any (loopless/simple) planar graph a proper coloring?

k-Color Theorem: Every planar graph is k-colorable.

There is no 3-color theorem since $\chi(K_4) = 4$ and K_4 is planar



Six-Color Theorem (Exercise 6.3.2): Every planar graph is 6-colorable.

Pf: Induction on $n(G)$.

Base: $n(G) \leq 6$. Can color every vertex a diff. color.

Inductive step: $n(G) > 6$.

By Thm 6.1.23a, $e(G) \leq 3n(G) - 6 < 3n(G)$.

Since by the deg. sum formula,

$$2e(G) = \sum_{v \in V(G)} d(v),$$

we must have a vertex $v \in V(G)$ of deg. < 6.

By the inductive hyp., $G \setminus v$ is 6-colorable, so take any proper 6-coloring of $G \setminus v$ and color v a diff. color from its neighbors to obtain a proper 6-coloring of G. $\square$

What about 5 colors? 4 colors?

Next time.