Announcements:

HW9 posted (due Wed. 11/29 9am)

No class or office hours Mon. 11/27

Remark 6.1.9:

a) G^* is always connected

b) If G is connected, $|V(G)| = |\{\text{faces of } G^*\}|$

and $|V(G^*)| = |\{\text{faces of } G\}|$

c) $(G^*)^* \cong G \iff G$ is connected

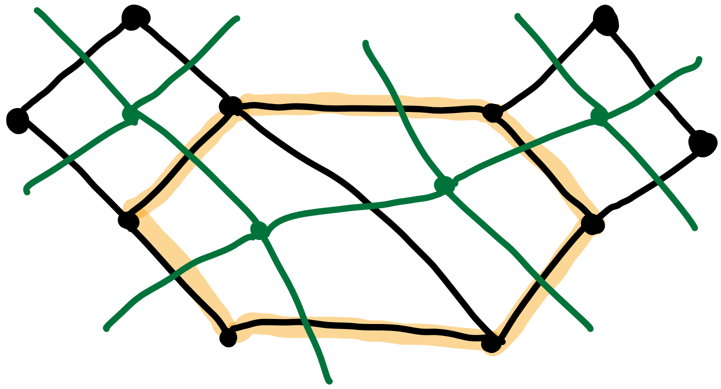
Pf: Homework!

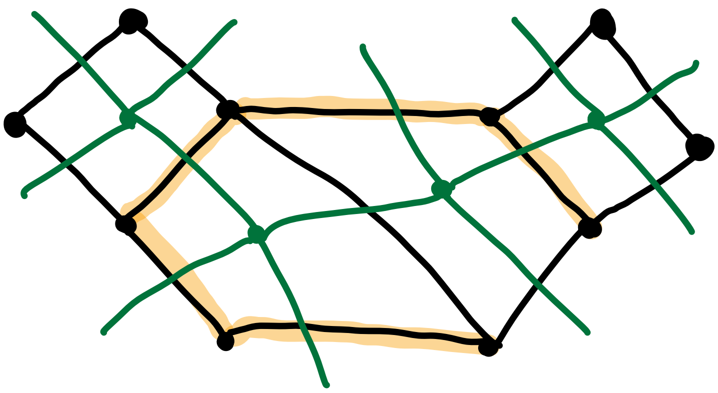
Thm 6.1.14: Let G be a connected graph.

Let $D \subseteq E(G)$, and let $D^* \in E(G^*)$ be the corresponding edges in G^*. Then,

D is the edge set of a cycle $\iff$ D^* is a minimal edge cut.

Pf: $\Rightarrow$ In G, D is the set of edges separating the faces inside D and the faces outside D, so in G^*, D^* is the set of edges from the vertices corresp. to the faces inside D to the verts. corresp. to the faces outside D; hence, D^* is an edge cut. To show that D^* is minimal, let C^* be a proper subset of D^* and let $e^* \in D^* \setminus C^*$, w/ corresp. edge $e \in E(G)$. Then, by the Jordan Curve Thm. (every curve has an inside and outside) applied to G, there is a path in G^* from any vertex to the vertex corresponding to the infinite face in G; hence $G \setminus C^*$ is connected, so D^* is minimal.





$\Leftarrow$: From the previous direction, if D^* is an edge cut, D must contain the edges of a cycle. If D contains othere edges, then removing them from D still contains a cycle, so D^* is not minimal.

$\square$

What can we say about the numbers of vertices $n := |V(G)|$, edges $e := |E(G)|$ and faces f of a planar graph?

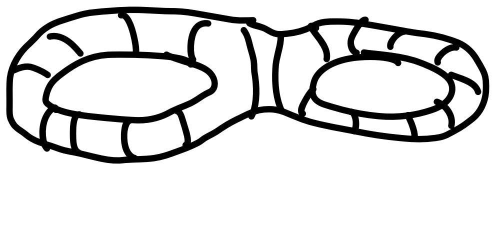
Euler's Formula: For any conn. planar graph G,

$$n - e + f = 2$$

Remark: This is the tip of the iceberg of one of the most important ideas in algebraic topology, called the Euler characteristic. For instance, for a graph drawn on a g-hole torus with no crossings,

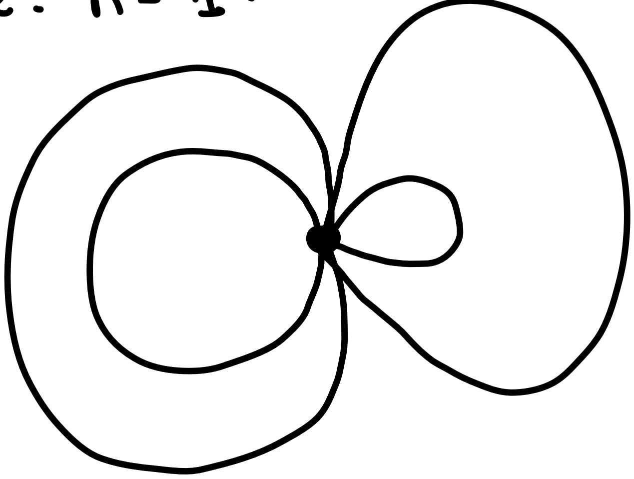
$$n - e + f = 2 - g$$

e.g. $g = 2$



Pf of Euler's formula: We use induction on n.

Base case: $n = 1$:



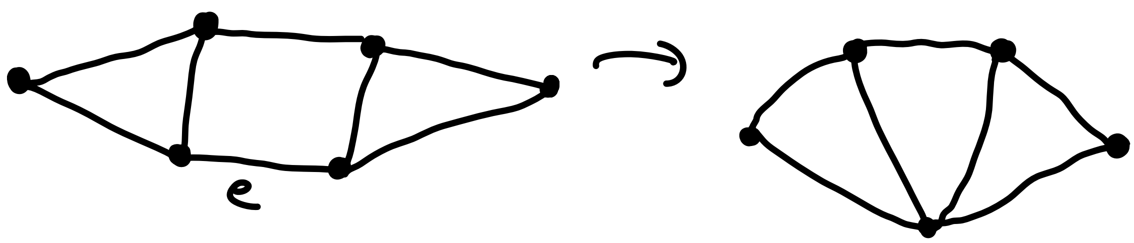
$f = e + 1$

$n - e + f$

$= 1 - e + e + 1 = 2$ ✓

Inductive step: $n > 1$.

Since G is conn., it has an edge that is not a loop. Contract along this edge to form the graph G'. Now, G' has $n-1$ vertices, $e-1$ edges, and f faces.



By the inductive hypothesis applied to G',

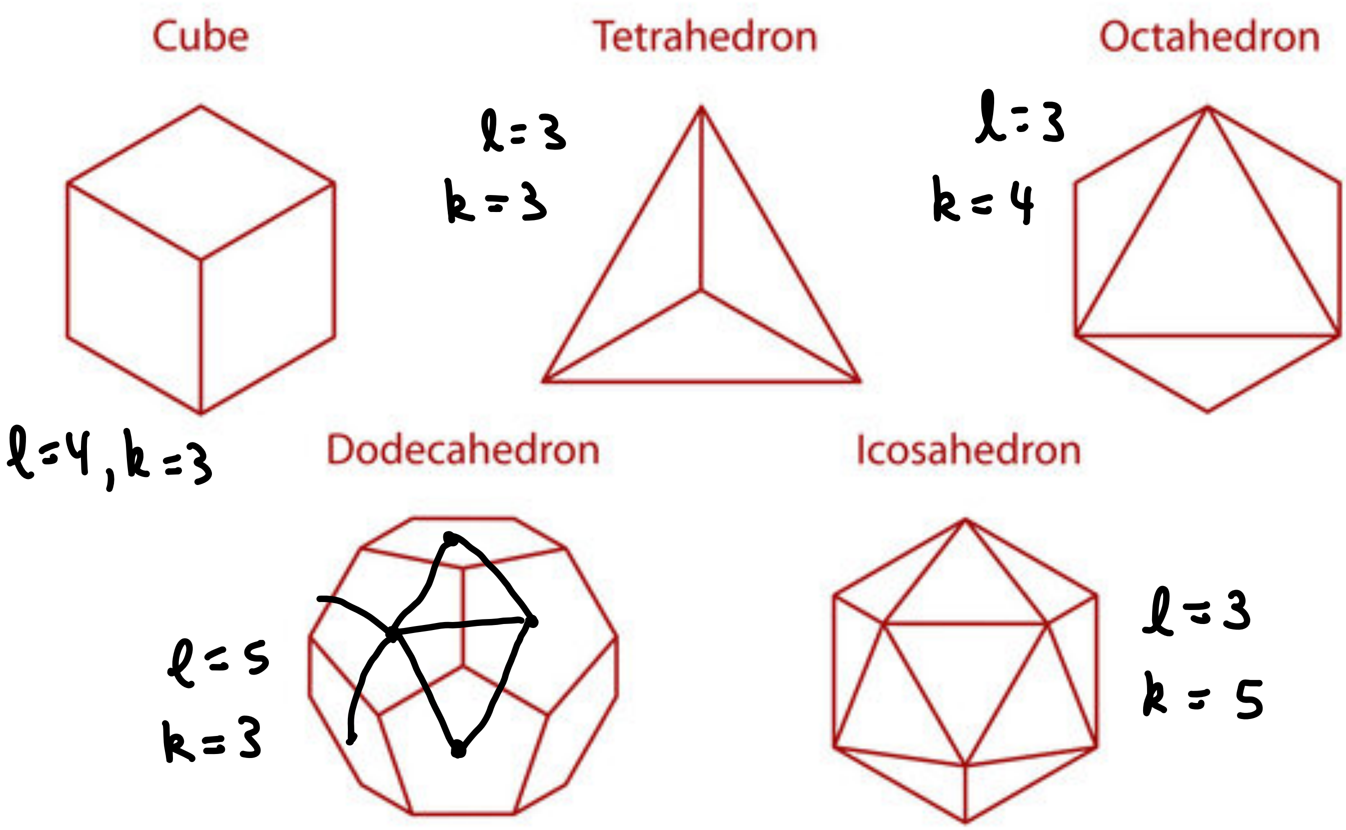
$(n-1) - (e-1) + f = 2$, so

$n - e + f = 2$ as well. $\square$

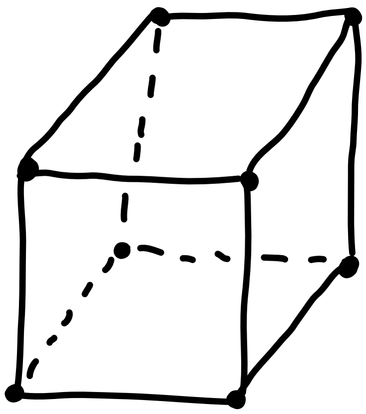
Application: regular polyhedra

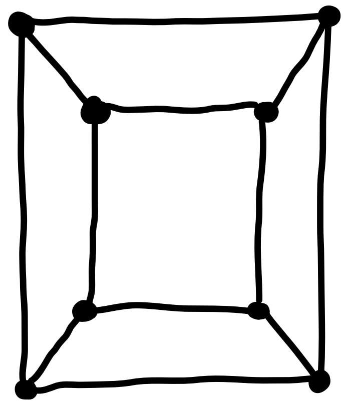
Def: A polyhedron is a 3D solid whose boundary consists of polygons, called faces. The edges/vertices are the edges/vertices of the polygons.

Def: A regular polyhedron is a solid whose boundary consists of identical regular polygons with the same number of faces around each vertex.



View these as a graph on a sphere, and "pull open" the back face to make a plane graph





8 vertices
12 edges
6 faces

$8 - 12 + 6 = 2$

Cor: Every polyhedron satisfies $n - e + f = 2$

Let's determine all the regular polyhedra:

n vertices
e edges
f faces

faces have l edges
vertices have k faces

Degree sum formula for G and G^*:

$$kn = 2e = \ell f$$

So by Euler's formula,

$$2 = n - e + f = \frac{2e}{k} - e + \frac{2e}{\ell} = e\left(\frac{2}{k} - 1 + \frac{2}{\ell}\right)$$

Since $e > 0$, $2 > 0$, $\frac{2}{k} - 1 + \frac{2}{\ell} > 0$, so

$\frac{2}{k} + \frac{2}{\ell} > 1$, so $2\ell + 2k > k\ell$,

so $(k-2)(\ell-2) < 4$.

Need $k, \ell \geq 3$ since dual of 2-reg. graph is not simple. Thus, $k, \ell \leq 5$.

Only possibilities for (k, ℓ) are:

$(3,3)$	$(3,4)$	$(3,5)$	$(4,3)$	$(5,3)$
tetrahedron	cube	dodecahedron	octahedron	icosahedron