Announcements

Quiz 3: this Friday in class (topics through today)

Midterm 3: Next Wed. 11/15 7:00-8:30pm Noyes 217

Recall: Def 5.2.1: Let G be a simple graph with $V(G) = \{v_1, \dots, v_n\}$. Let $U = \{u_1, \dots, u_n\}$.

<u>Mycielski's construction</u> gives a graph $G' := Myc(G)$ with

$$V(G') = V(G) \sqcup U \sqcup \{w\}$$

$$E(G') = E(G) \sqcup \{u_i v \mid 1 \leq i \leq n, v \in N(v_i)\} \sqcup \{u_i w \mid 1 \leq i \leq n\}$$

Thm 5.2.3: For all $k \geq 1$, there exists a triangle-free graph G with $\chi(G) = k$.

Pf: We show that if G is a simple Δ-free graph, $G' := Myc(G)$ is a simple Δ-free graph w/

$$\chi(G') = \chi(G) + 1 \qquad (k := \chi(G))$$

Δ-free: Any Δ in G' must contain a vertex in U since G is Δ-free and $N(w) = U$. However, since U is an indep. set, the Δ must contain two vertices in $V(G)$ and some $u_i \in U$. But, since $N(v_i) \cap V(G) = N(u_i) \cap V(G)$, replacing u_i by v_i still creates a Δ, a contradiction.

$\chi(G') \leq k+1$: If f is a proper k-coloring of G, then $g(v_i) = g(u_i) = f(v_i)$, $g(w) = k+1$ gives a proper $(k+1)$-coloring of G'.

$\chi(G') > k$: Assume for a contradiction that g is a proper k-coloring of G'. WLOG, say that $g(w) = k$; then $g(u_i) \leq k-1$ $\forall i$. Let

$$A = \{v \in G \mid g(v) = k\}$$

and let f be the k-coloring of G' where

$$f(x) = \begin{cases} g(x), & \text{if } x \notin A \\ g(u_i), & \text{if } x \in A,\ x = v_i \end{cases}$$

$f|_G$ is a $(k-1)$-coloring of G, so the result follows by contradiction if we can show $f|_G$ is proper. No two vertices of A are adjacent since their colors in g are the same, so the only edges which could violate properness are of the form $v_i v$, $v_i \in A$, $v \in V(G) \setminus A$. However, since v is adjacent to v_i in G', it is also adjacent to u_i, so

$$f(v) = g(v) \neq g(u_i) = f(v_i),$$

and therefore $f|_G$ is proper. $\square$

Let's summarize results so far about $\chi(G)$. Our upper-bound results involve vertex degrees.

- $\chi(G) \le n(G)$
- $\chi(G) \le 1 + \Delta(G)$, and "usually", $\chi(G) \le \Delta(G)$
- $\chi(G) \le 1 + \max_i \min\{d_i, i-1\}$
- $\chi(G) \le 1 + \max_{H \subseteq G} \delta(H)$

Meanwhile,

- $\chi(G) \ge \omega(G)$, and potentially $\chi(G) \gg \omega(G)$

So if we allow many vertices and high degrees, are we forced to accept (potentially) high chromatic number?

We'll come back to this question soon with regards to <u>planar graphs</u>.

First, a detour to some counting problems...

Def 5.3.1: Let G be a graph and $k \in \mathbb{N}$.

a) $\chi(G;k)$ is the number of proper colorings

$f: V(G) \to \{1, \dots, k\}$ of G w/ k colors.

e.g. If $k < \chi(G)$, $\chi(G;k) = 0$ and
if $k \geq \chi(G)$, $\chi(G;k) \geq 1$

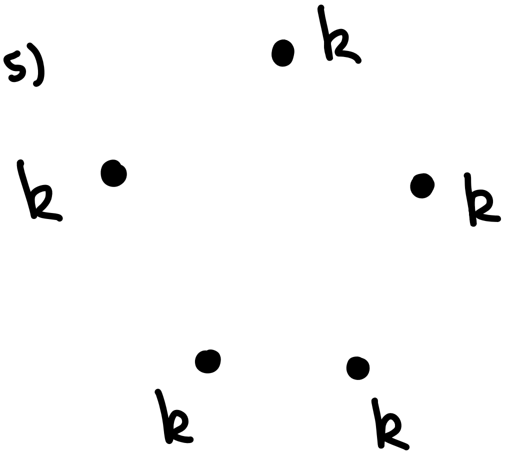
b) If we think of $\chi(G;k)$ as a function of k, we call $\chi(G;k)$ the <u>chromatic polynomial</u> of G.

need to justify this

Class activity:

a) Find $\chi(\overline{K_n}; k)$ as a function of k

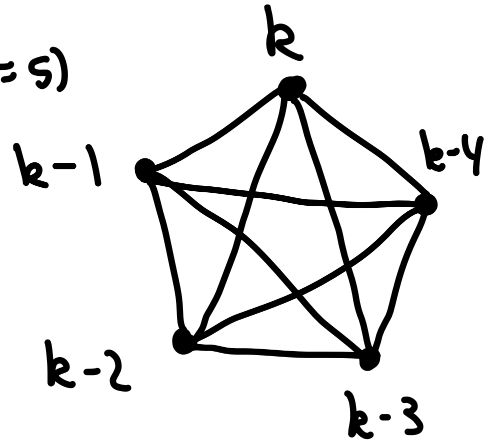
$(n=5)$



$$\chi(\overline{K_n}; k) = k^n$$

b) Find $\chi(K_n; k)$ as a function of k

(n=5)



$$\chi(K_n; k) = \binom{k}{n} n!$$

$$= \frac{k!}{n!(k-n)!} n! = \frac{k!}{(k-n)!}$$

$$= k(k-1)\cdots(k-n+1)$$

$$=: k_{(n)}$$

Prop 5.3.4: $\chi(G, k)$ is a polynomial in k. In particular,

$$\chi(G; k) = \sum_{r=1}^{n(G)} p_r(G)\, k_{(r)}$$

where $p_r(G)$ is the number of ways to write $V(G)$ as a disjoint union of r nonempty indep. sets and $k_{(r)} := k(k-1)\cdots(k-r+1)$

Pf: Every proper coloring is det'd uniquely by the following choices:

Choice 1: r, $1 \le r \le n(G)$

Choice 2: Choose one of the $p_r(G)$ ways to write $V(G) = V_1 \sqcup \cdots \sqcup V_r$ where $V_1, \ldots, V_r$ are nonempty indep. sets

Choice 3: Choose one of the k colors for V_1, then one of the remaining $k-1$ colors for V_2, etc.

Thus,

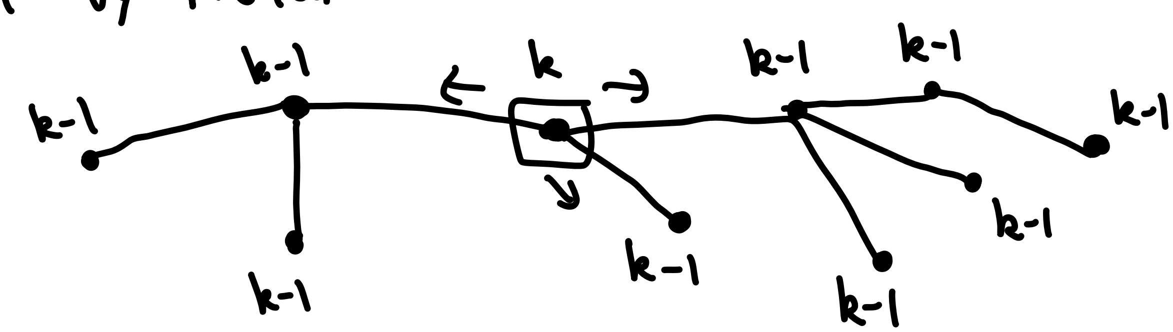
$$\chi(G;k) = \sum_{r=1}^{n(G)} p_r(G) \cdot \underbrace{k(k-1)\cdots(k-r+1)}_{k_{(r)}},$$

a poly. in k. $\square$

Prop 5.3.3: If T is a tree w/ n vertices, then

$$\chi(G;k) = k(k-1)^{n-1}$$

Pf by picture:



$\square$

Remark: $\chi(G)$ is the smallest nonnegative integer a s.t.

$$k-a \nmid \chi(G;k)$$

There is a method to compute $\chi(G;k)$ recursively using deletion-contraction, allowing for a computation of $\chi(G;k)$, and thus $\chi(G)$, for any (individual) graph G.

Thm 5.3.6: Let G be a simple graph and $e \in E(G)$. Then,

$$\chi(G;k) = \chi(G \setminus e; e) - \chi(G \cdot e; k)$$

Pf: Next time