Announcements

No class Monday 11/27 (day after Fall Break)

Quiz 3: Fri. 11/10 in class

Midterm 3: Wed. 11/15 7:00-8:30pm Noyes 217

We already know using greedy coloring that

$$\chi(G) \le \Delta(G) + 1$$

And equality is possible.

$$\chi(K_n) = n = \Delta(K_n) + 1$$

$$\chi(C_{2k+1}) = 3 = \Delta(C_{2k+1}) + 1$$

Brooks' Thm (5.1.22): If G is connected and G is <u>not</u> a complete graph or odd cycle, then

$$\chi(G) \le \Delta(G).$$

Pf: Let G be a connected graph, and let $k = \Delta(G)$.

If $k=0$, $G = K_1$

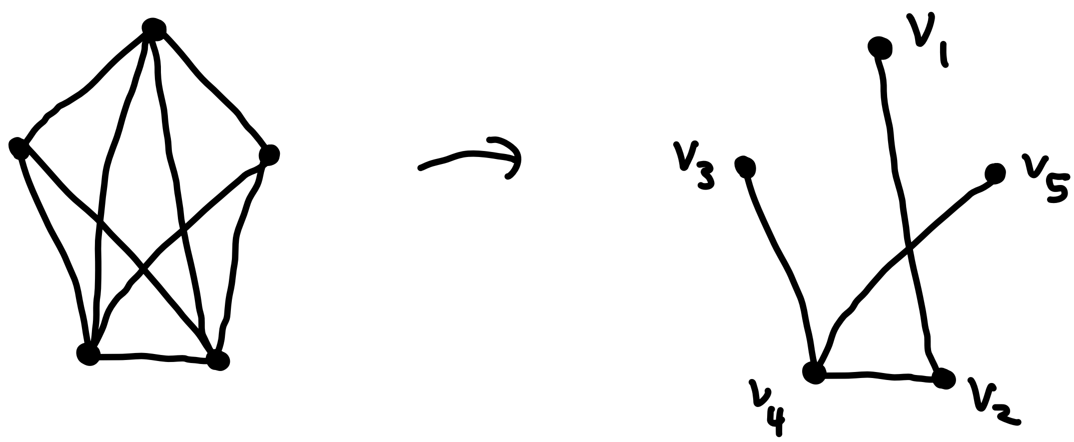
If $k=1$, $G = K_2$

If $h=2$, G is an odd cycle or an even cycle or a path.
In the latter two cases, G is bipartite, so $\chi(G)=2 = \Delta(G)$.

Assume $k \geq 3$, and use greedy coloring. Want every vertex to have $\leq k-1$ lower-index neighbors.

Case 1: G is not k-regular. Choose $v_n \in V(G)$ s.t. $d(v_n) < k$. Let T be a spanning tree of G and choose an ordering of $V(G) = \{v_1, \ldots, v_n\}$ s.t.

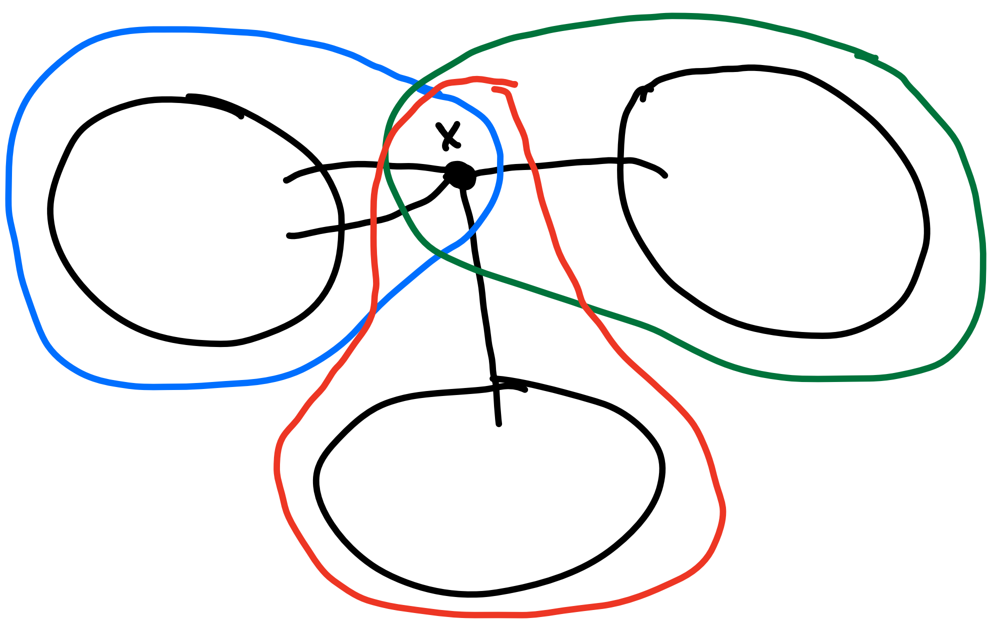
$$i < j \iff d_T(v_i, v_n) \geq d_T(v_j, v_n)$$



G T

Hence, every vertex other than v_n has ≥ 1 higher-indexed neighbor, so every vertex has $\leq k-1$ lower-indexed neighbors.

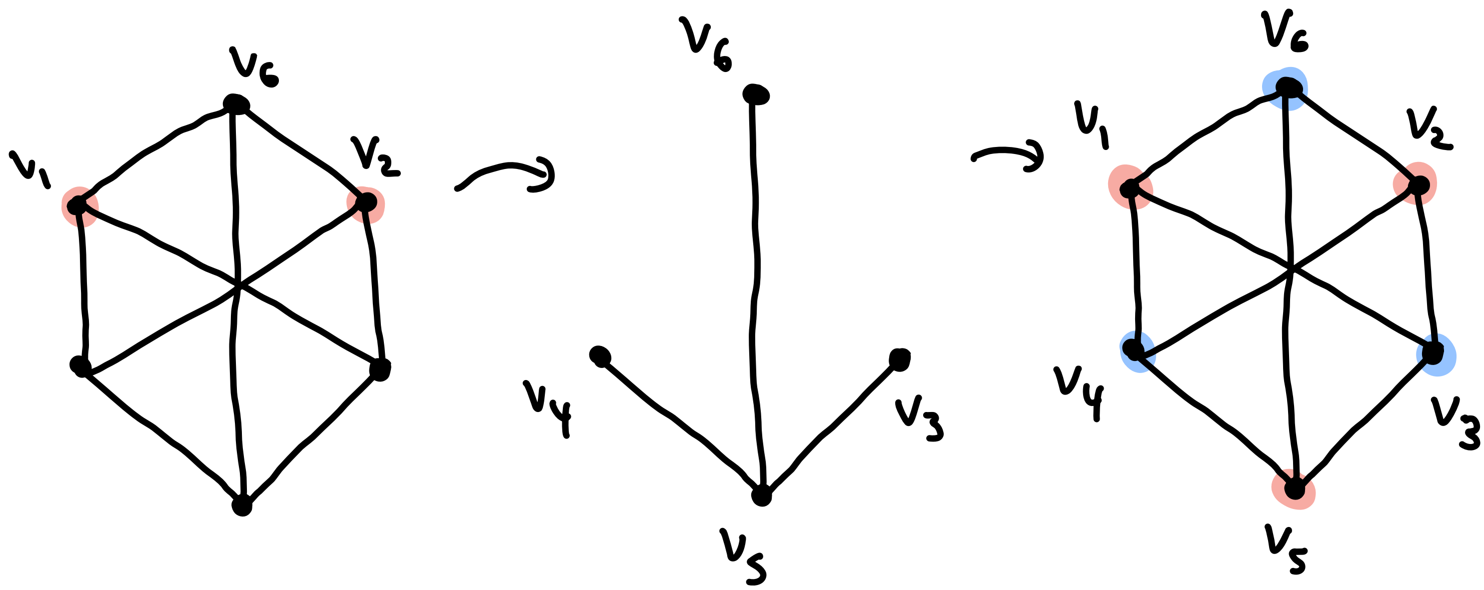
Case 2: G is k-regular and has a cut vertex x. Then we use the previous method on all components of $G \setminus x$, along w/ x, and, permuting colors if necessary, this gives a proper coloring of G.



Case 3: G is k-regular and 2-connected.

First, suppose that $\exists v_n \in V(G)$ s.t. $N(v_n) \ni v_1, v_2$ w/ v_1 and v_2 not adjacent and $G \setminus \{v_1, v_2\}$ is connected. Then, color v_1 and v_2 the same color, and use the spanning tree trick on $G \setminus \{v_1, v_2\}$. Every vertex

$v_3, \ldots, v_{n-1}$ has $\le k-1$ earlier neighbors, while v_n has ≥ 2 neighbors of the same color.



Finally, we show that every k-reg. 2-conn. graph G has such vertices. Let $x \in V(G)$. Since G is 2-conn., $G \setminus x$ is conn. If $G \setminus x$ is 2-conn, let

$v_1 = x$

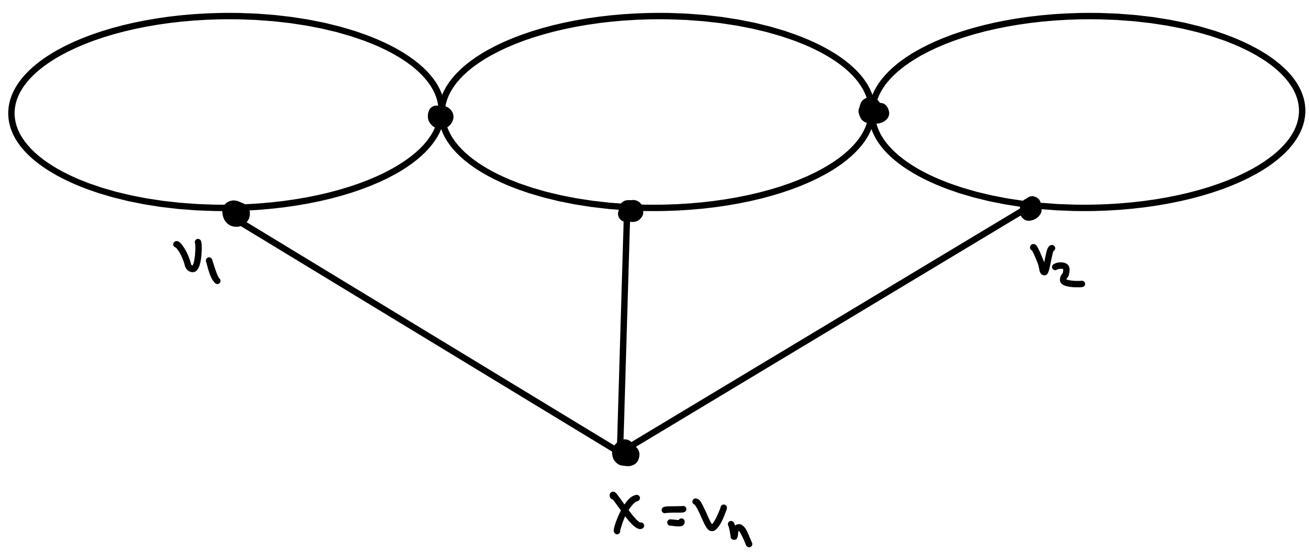
v_2 have distance 2 from x (exists since G is reg., not complete)

v_n be a common neighbor of v_1, v_2

$G \setminus \{v_1, v_2\} = G \setminus \{x, v_2\}$ is conn. ✓

If $G \setminus x$ is not 2-conn., let $H_1, H_2, \ldots, H_j$

be the maximal connected subgraph of $G \setminus x$ with no cut-vertices ("blocks"). These subgraphs may contain more than one cut vertex, but at least two, say H and H', contain exactly one ("leaf blocks")



Let $v_1 \in H \cap N(x)$

$v_2 \in H' \cap N(x)$

$v_n = x$

Then $G \setminus \{x, v_1, v_2\}$ is conn., and since $d_G(x) \geq 3$, $G \setminus \{v_1, v_2\}$ is conn. $\square$

Last time: showed that for all G, $\chi(G) \geq \omega(G)$, and for interval graphs, $\chi(G) = \omega(G)$.

Turns out $\chi(G)$ can be way bigger than $\omega(G)$. In fact,

Thm 5.2.3: For all $k \geq 1$, there exists a triangle-free graph G with $\chi(G) = k$.

Def 5.2.1: Let G be a simple graph with $V(G) = \{v_1, \dots, v_n\}$. Let $U = \{u_1, \dots, u_n\}$. Mycielski's construction gives a graph $G' := Myc(G)$ with

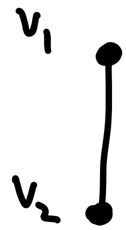
$$V(G') = V(G) \sqcup U \sqcup \{w\}$$

$$E(G') = E(G) \sqcup \{u_i v \mid 1 \leq i \leq n, v \in N(v_i)\} \sqcup \{u_i w \mid 1 \leq i \leq n\}$$

Class activity: Find

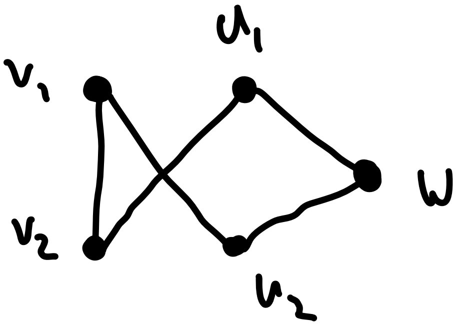
a) $Myc(K_2)$

b) $Myc(Myc(K_2))$



K_2

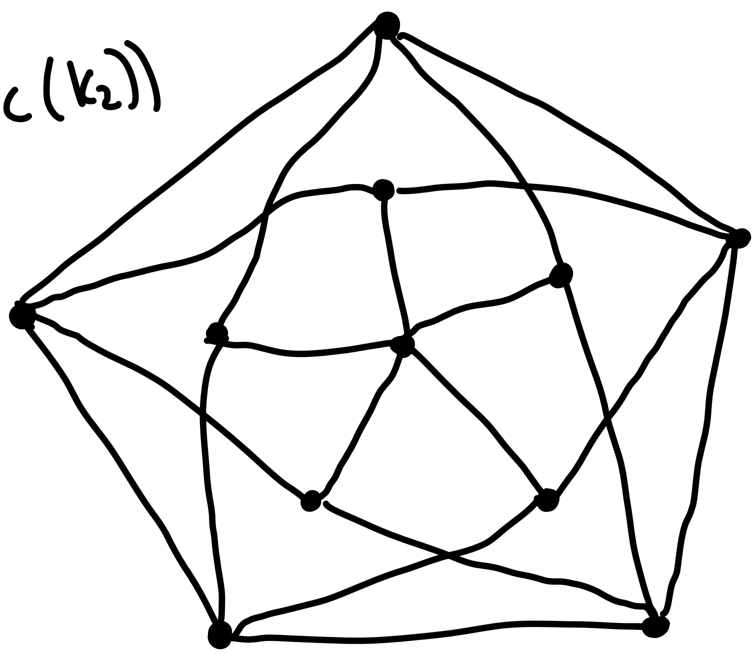
$\chi = 2$



$Myc(K_2)$

$\chi = 3$

$Myc(Myc(K_2))$



$\chi = 4$

Pf of Thm 5.2.3: