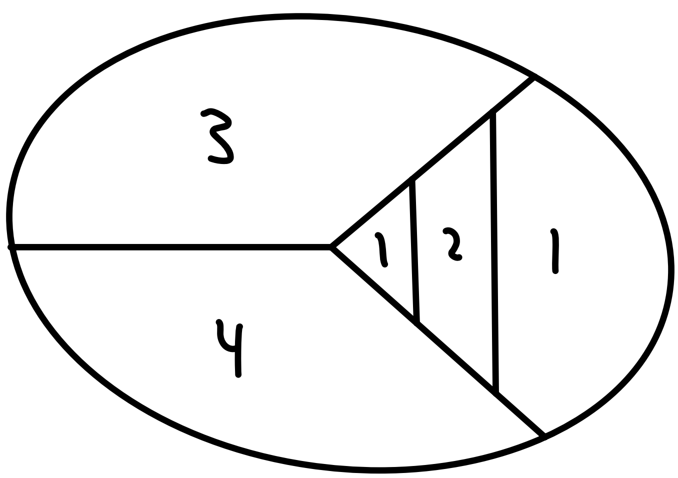
Announcements:

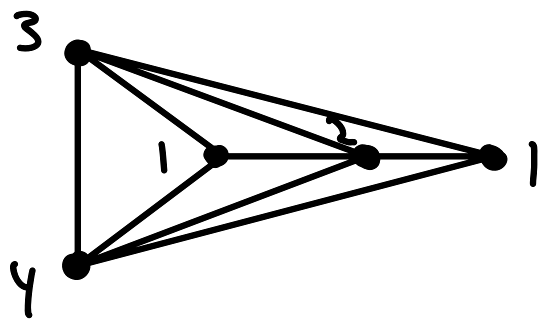
Grading up to date / released

Quiz 3: Fri. 11/10 in class

Midterm 3: Wed. 11/15 7:00-8:30pm Noyes 217

Chapter 5: Coloring of graphs





Def 5.1.1:

a) A k-coloring of a graph G is a labeling $f: V(G) \to S$ where $|S| = k$ (usually $S = \{1, \dots, k\}$)

The elements of S are called <u>colors</u>

b) A k-coloring is called proper if adjacent vertices have different labels. In this case, we call G k-colorable

c) The chromatic number of G is

"chi" $\chi(G) :=$ least k s.t. G is k-colorable

We call G k-chromatic if $\chi(G) = k$

Remark 5.1.2:

a) If G has a loop, $\chi(G) = \infty$, so we assume G is loopless

b) k-colourable $\Leftrightarrow$ k-partite $\Leftrightarrow$ $V(G)$ is the union of k indep. sets

Def 5.1.4: If $\chi(G) = k$ and every proper subgraph H of G has, $\chi(H) < k$, then G is color-critical or k-critical

Ex: $\chi(G) = 1 \Leftrightarrow G$ has no edges, so K_1 is the only 1-critical graph

Class activity:

a) Characterize 2-critical graphs

b) Characterize 3-critical graphs

odd cycles

c) Find a color-critical graph w/ chromatic number larger than 3.

K_n is n critical

Main goal for the rest of this course:

Compute or bound $\chi(G)$ for different classes of graphs.

Recall: $\alpha(G)$ = largest size of independent set

Let $\omega(G) :=$ largest size of clique

Easy bounds (5.1.7): For all (loopless) graphs G,

a) $\chi(G) \leq n(G)$

b) $\chi(G) \geq \omega(G)$

c) $\chi(G) \geq \dfrac{n(G)}{\alpha(G)}$

d) If $H \subseteq G$, $\chi(H) \leq \chi(G)$ $\square$

Greedy coloring algorithm:

Start: order $V(G) = \{v_1, \dots, v_n\}$

For $i = 1, 2, \dots, n$:

Color v_i the smallest color not already used by its neighbors

Prop 5.1.13: $\chi(G) \leq \Delta(G)+1$

Pf: Greedy coloring can use at most $\Delta(G)+1$ colours since each vertex has $\leq \Delta(G)$ neighbors. $\square$

Can do better:

Prop 5.1.14: If G has degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, then

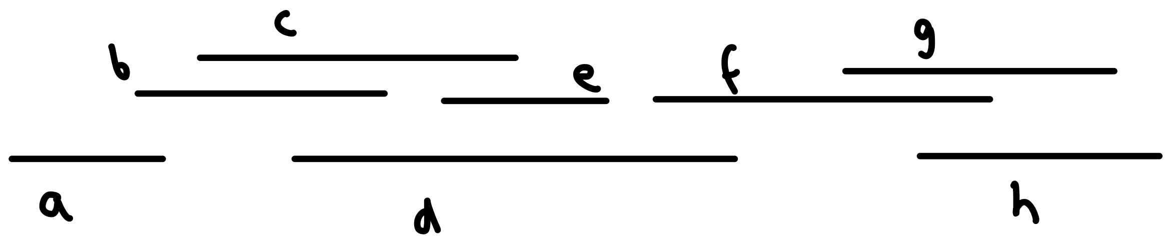
$$\chi(G) \leq 1 + \max_i \min\{d_i, i-1\} \leq \Delta(G)$$

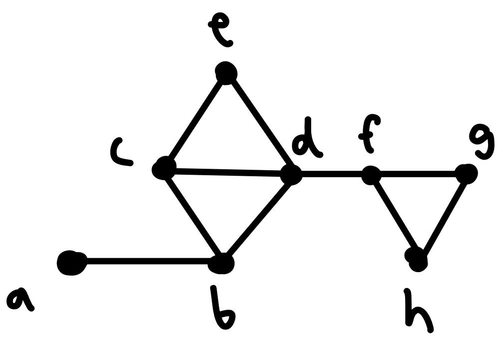
Pf: Again, use greedy coloring. Order the vertices s.t. their degree is weakly decreasing. Then each vertex has at most $\min\{d_i, i-1\}$ earlier neighbors. $\square$

Recall that $\chi(G) \geq \omega(G)$.

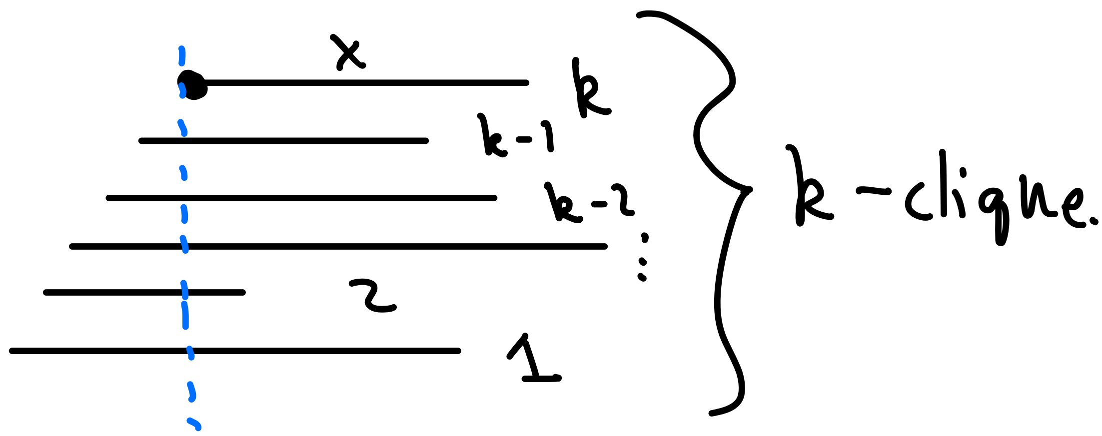
Prop 5.1.16: If G is an <u>interval graph</u>, $\chi(G) = \omega(G)$.

Def: An <u>interval graph</u> is a graph which has an <u>interval representation</u>, an interval in $\mathbb{R}$ for each $v \in V(G)$ s.t. v and w are adjacent iff the corresp. intervals overlap.





Pf of Prop 5.1.16: Order the vertices according to the left endpoints of the interval. We show that the greedy coloring alg. produces a $\omega(G)$-coloring. Let k be the max. color assigned by greedy coloring, and suppose x receives color k.



$\omega(G) = k \geq \chi(G)$, so they're equal. $\square$

Lemma 5.1.18: If H is k-critical, then $\delta(H) \geq k-1$.

Pf: Let $x \in V(G)$. Sinze H is k-crit., $H \setminus x$ is $(k-1)$-colorable. If $d_H(x) < k-1$, then we can take a $(k-1)$-coloring of $H \setminus x$ and assign x any of the colors not assigned to its neighbors to obtain a $(k-1)$-coloring of H, a contradiction. $\square$

Cor (Thm 5.1.19): $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$

Pf: Let H' be a $\chi(G)$-crit. subgraph of G.
Then, by Lemma 5.1.18,

$$\chi(G) - 1 = \chi(H') - 1 \leq \delta(H') \leq \max_{H \subseteq G} \delta(H) \quad \square$$

We already know using greedy coloring that

$$\chi(G) \leq \Delta(G) + 1$$

And equality is possible.

$$\chi(K_n) = n = \Delta(K_n) + 1$$

$$\chi(C_{2k+1}) = 3 = \Delta(C_{2k+1}) + 1$$

Brooks' Thm (5.1.22): If G is connected and G is <u>not</u> a complete graph or odd cycle, then

$$\chi(G) \leq \Delta(G).$$

Pf: Next time