Announcements:

HW8 posted (due Wed. 11/8)

Quiz 3: Fri. 11/10 in class

Midterm 3: Wed. 11/15 7:00-8:30pm Noyes 217

Recall Cor 4.3.8: Let N be a network. If f is a feasible flow and $[S,T]$ is a source-sink cut, then

$$\mathrm{val}(f) \le \mathrm{cap}(S,T)$$

Implication: $\max_f \mathrm{val}(f) \le \min_{[S,T]} \mathrm{cap}(S,T)$

Max-flow, min-cut theorem (4.3.11):

$$\max_f \mathrm{val}(f) = \min_{[S,T]} \mathrm{cap}[S,T]$$

Remark: This result has connections to Menger's Thm., Halls Thm.*, etc. *see homework

Pf. idea: If $\mathrm{val}(f) < \min_{[S,T]} \mathrm{cap}[S,T]$, find an f-augmenting path.

Ford-Fulkerson algorithm:

Input: A feasible flow f in a network N

Start: $\underbrace{R=\{s\}}_{\text{"reached"}}$, $\underbrace{S=\emptyset}_{\text{"searched"}}$, $\underbrace{\Pi=\{\pi_s := s\}}_{\text{paths in underlying graph}}$

While $R \neq S$ and $t \notin R$:

  Let $v \in R \setminus S$

  For all $vw \in E(N)$:

    If $f(vw) < c(vw)$ and $w \notin R$:

      Add w to R

      Add $\pi_w := \pi_v, w$ to Π

  For all $uv \in E(N)$:

    If $f(uv) > 0$ and $u \notin R$:

      Add u to R

      Add $\pi_u := \pi_v, u$ to Π

  Add v to S

If $t \in R$:

  Output π_t (f-augmenting path)

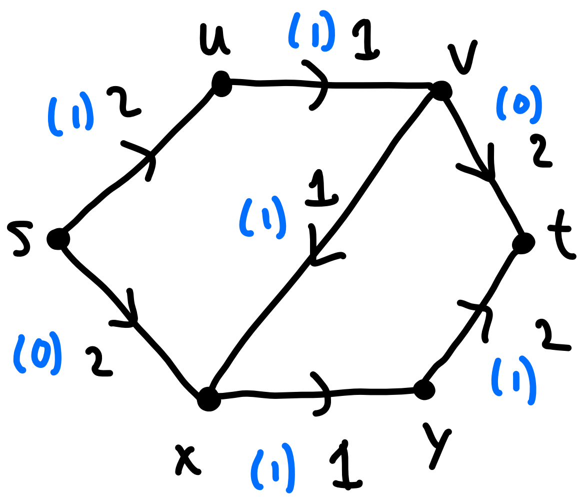
Otherwise (i.e. $R=S$):

  Output $[S, \bar{S}]$ (cut w/ capacity val(f))

If Ford-Fulkerson returns an f-augmenting path, can augment along the path, and rerun.

Class activity: Run FF on the following graph repeatedly, and obtain a max. flow and min. cut

f:



R: s, u, x, v, t

S: s, u, x

Π: $\Pi_s = s$

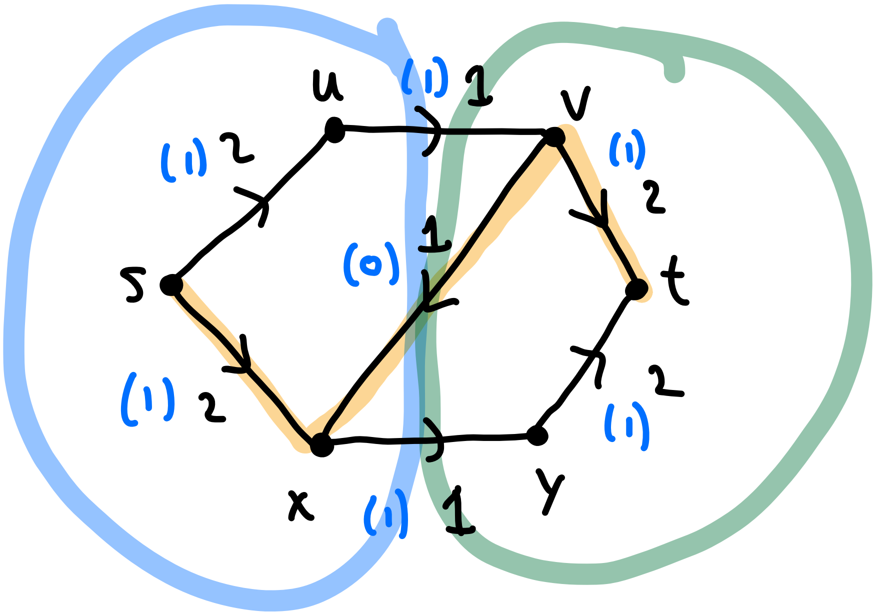
$\Pi_u = s, u$

$\Pi_x = s, x$

$\Pi_v = s, x, v$

$\underbrace{\Pi_t = s, x, v, t}_{\text{f-aug. path}}$

f':



R: s, u, x

S: s, u, x

Π: $\Pi_s = s$

$\Pi_u = s, u$

$\Pi_x = s, x$

f' is a maximum flow: $val(f') = 2$

$[S, \overline{S}]$ is a min. cut: $cap(S, \overline{S}) = 2$

Pf of max-flow, min-cut theorem when $c(e) \in \mathbb{Q}_{\geq 0}$:

For every network N, the zero flow ($f(e)=0\ \forall e$) is feasible. Given a feasible flow, apply the FF algorithm. Since every iteration adds a vertex to S, the alg. must terminate.

We have two cases.

Case 1: If $t \in R$, then $\pi_t \in \Pi$ is an s,t-path in the underlying graph, and in each step on the path, the flow can be increased or decreased as needed. Thus, π_t is an f-aug. path.

Case 2: If $R=S$, then $[S,\overline{S}]$ is a source-sink cut, and since no vertices of $T := \overline{S}$ were added to S, every edge $e \in [S,T]$ has $f(e)=c(e)$ and every edge $e \in [T,S]$ has $f(e)=0$, so $f^+(S) = \mathrm{cap}(S,T)$ * and $f^-(S)=0$*. Therefore,

$$\mathrm{val}(f) \underset{\text{last time}}{=} f^+(S) - f^-(S) = \mathrm{cap}(S,T) - 0 = \mathrm{cap}(S,T).$$

* shifted

So f is maximum and $[S,T]$ is minimum by Cor. 4.3.8.

Therefore, every maximum flow has a (minimum) cut of the same size.

All that remains is to show that a maximum flow always exists. Let a be the lcm of the denoms. of all capacities in lowest terms. By algebra, every augmentation is at least $1/a$, so after at most $a \sum_{e \in E(N)} c(e)$ augmentations, we must arrive at a maximum flow (and minimum cut). $\square$