Network flow:

Def 4.3.1/4.3.2:

a) A network is a digraph N w/

- A source vertex s
- A sink vertex t
- A nonneg. capacity $c(e)$ for each edge e

b) A flow f is an assignment of a value $f(e)$ to every edge e.

c) Let

$$f^+(v) = \sum_{\underset{v}{\bullet}\to\bullet} f(e) \qquad\qquad f^-(v) = \sum_{\bullet\to\underset{v}{\bullet}} f(e)$$

"outflow" "inflow"

d) f is feasible if

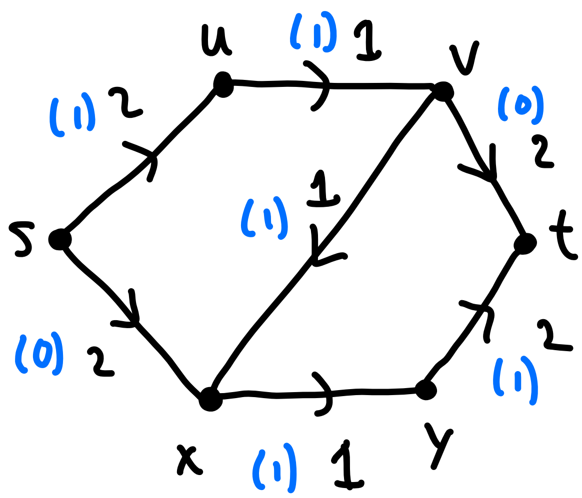
- $0 \leq f(e) \leq c(e)$ for all $e \in E(N)$ (capacity constraints)
- $f^+(v) = f^-(v)$ for all $v \in V(N) \setminus \{s,t\}$ (conservation constraints)

e) The value of f is $\mathrm{val}(f) := f^-(t) - f^+(t)$

f) A maximum flow is a feasible flow of maximum value

Class activity: Is this flow feasible? Maximum?
What is its value?

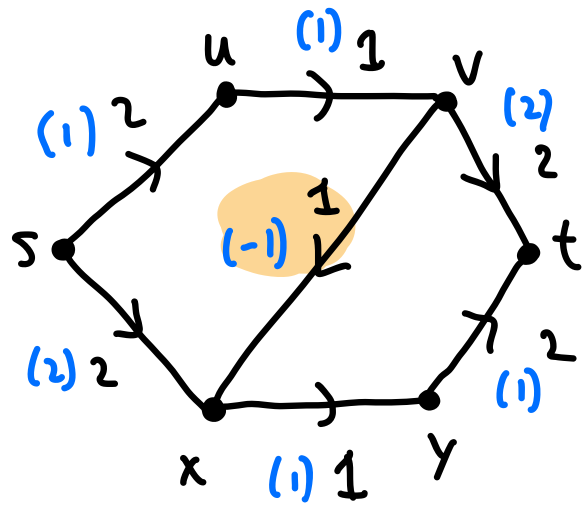
a)



Feasible

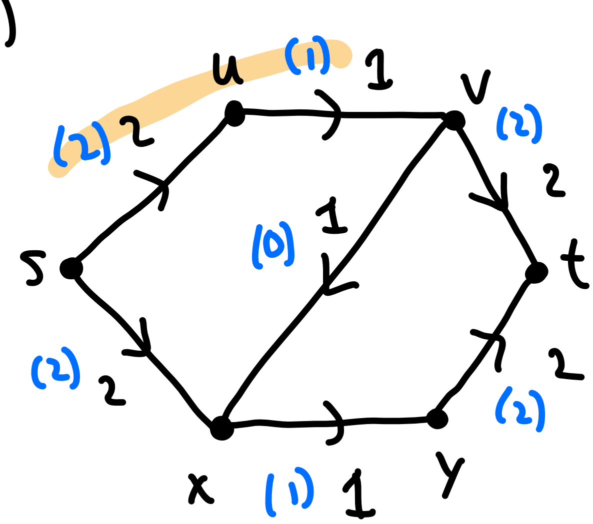
Value = 1

b)



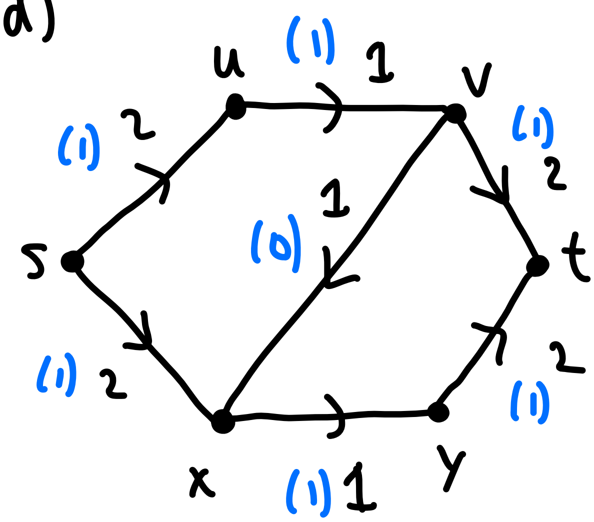
Not feasible

c)



Not feasible

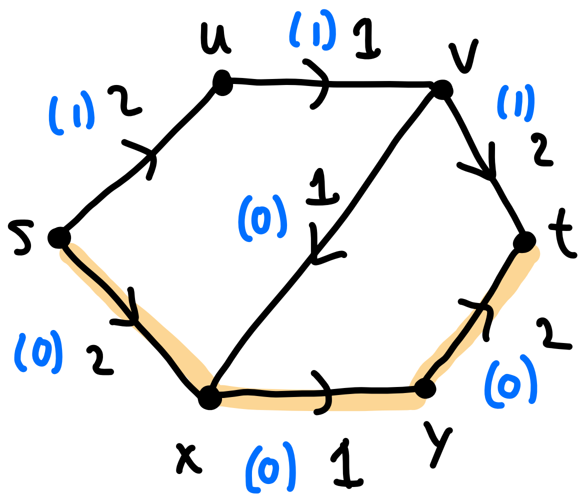
d)

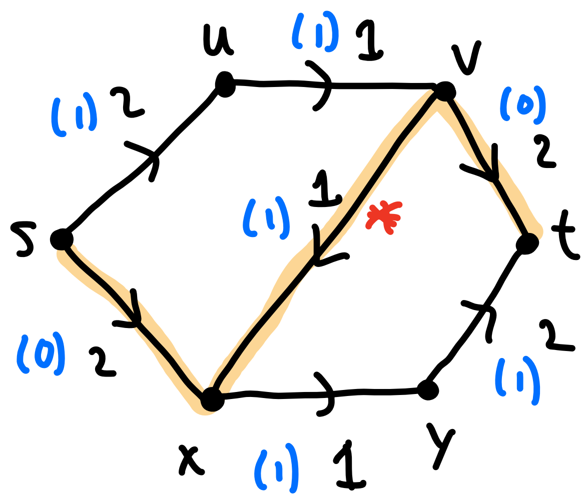


Feasible

Value = 2

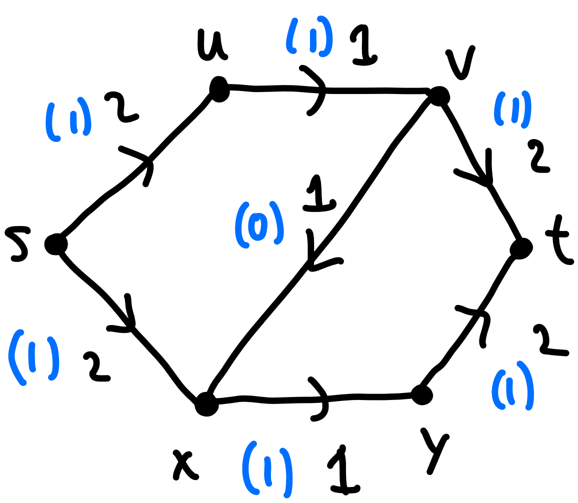
When we have an s,t-path with extra capacity...

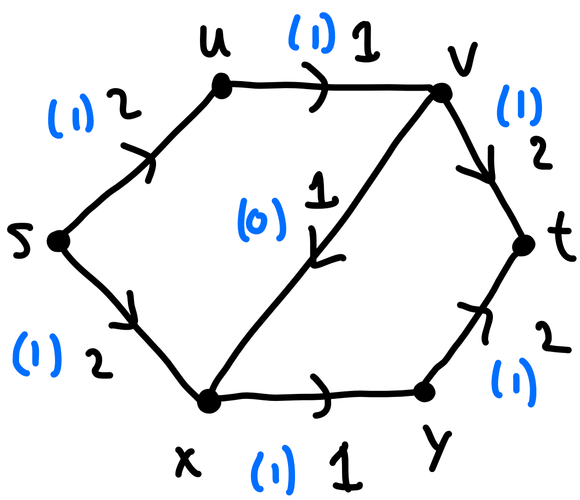




* need to consider more than just paths

we can <u>augment</u> the flow





Def 4.3.4: Let f be a feasible flow in a network N.

a) An <u>f-augmenting path</u> is an s,t-path in the underlying graph G s.t.

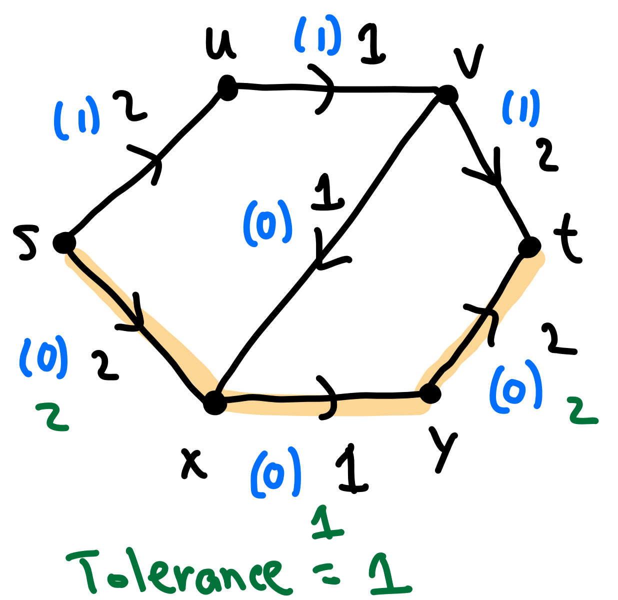
- If P follows e forwards, then $f(e) < c(e)$
- If P follows e backwards, then $f(e) > 0$

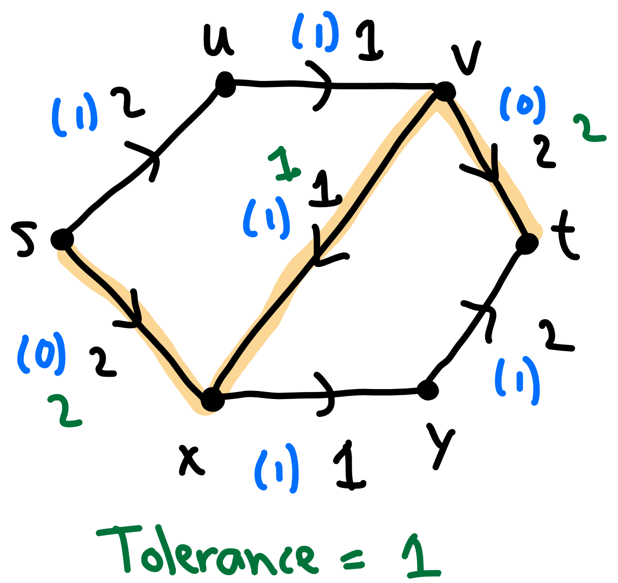
b) If $e \in E(P)$, then

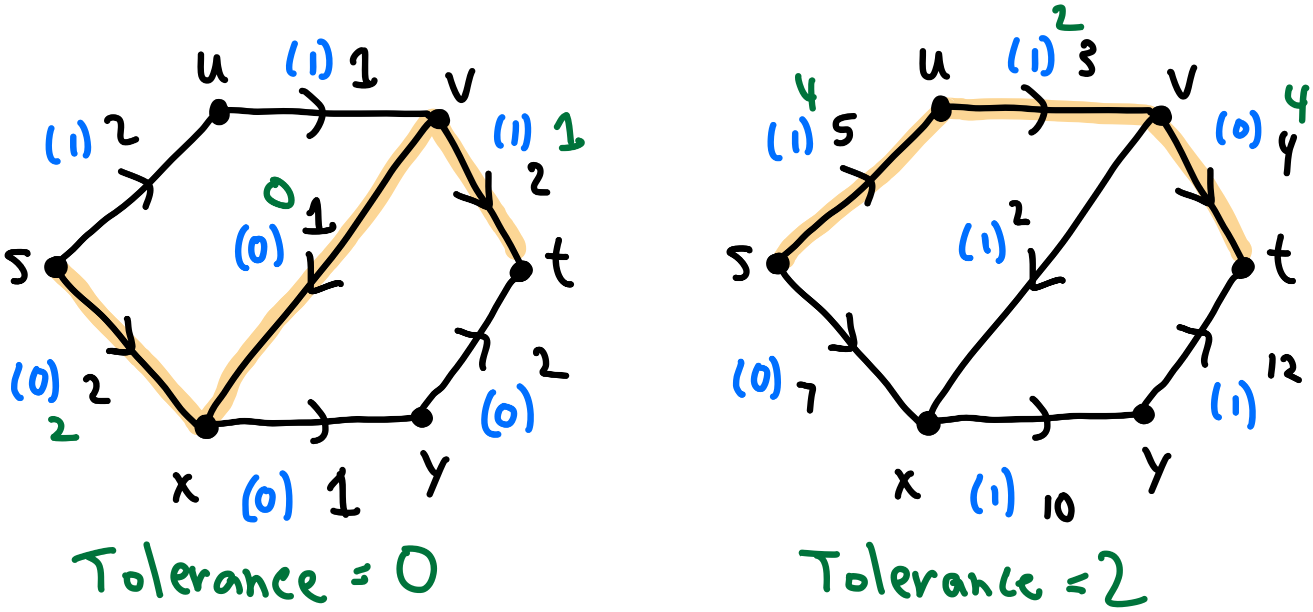
$$\varepsilon(e) = \begin{cases} c(e) - f(e), & \text{if } P \text{ follows } e \text{ forwards} \\ f(e) - 0, & \text{if } P \text{ follows } e \text{ backwards} \end{cases}$$

c) The tolerance of P is $\min_{e \in E(P)} \varepsilon(e)$.

Class activity: find the tolerances







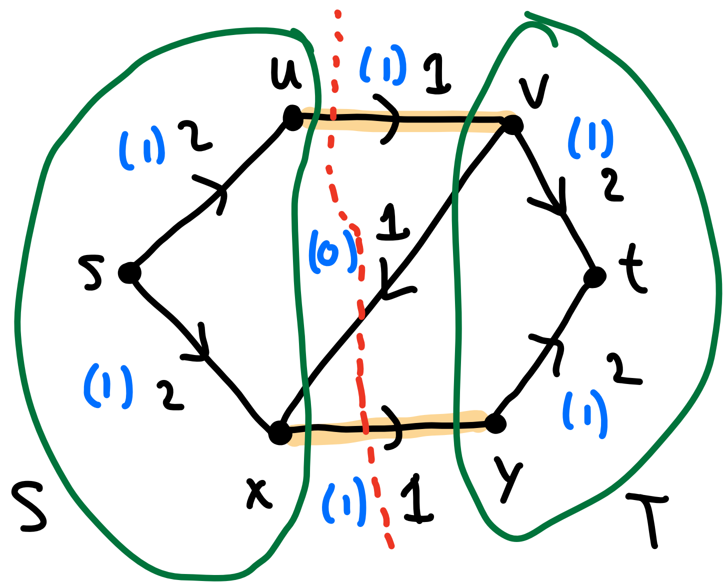
Lemma 4.3.5: If P is an f-augmenting path w/ tolerance z, then the flow

$$f'(e) = \begin{cases} f(e) + z, & \text{if } P \text{ follows } e \text{ forwards} \\ f(e) - z, & \text{if } P \text{ follows } e \text{ backwards} \\ f(e), & \text{if } e \notin P \end{cases}$$

is feasible with value $\text{val}(f') = \text{val}(f) + z$

Pf: Check capacity constraints, conservation constraints, and compute value of flow into t. $\square$

Now, how can we show a flow is maximum?



● $= [S, T]$

$$\begin{aligned} cap(S,T) &= c(uv) + c(xy) \\ &= 1 + 1 = 2 \end{aligned}$$

There's a "bottleneck" from the left half to the right half.

Def 4.3.6:

a) A <u>source/sink cut</u> $[S,T]$ is an $\underbrace{\text{edge cut}}_{\text{all edges } \underline{\text{from}}\ S\ \underline{\text{to}}\ T}$ where $s \in S$, $t \in T$, and $\overline{S} = T$.

b) The capacity of $[S,T]$ is

$$cap(S,T) := \sum_{e \in [S,T]} c(e)$$

c) If $U \subseteq V(N)$, let

$$f^+(U) = \sum_{\underset{U}{\bullet} \to \bullet} f(e) = \sum_{v \in U} f^+(v)$$

$$f^-(U) = \sum_{\bullet \to \underset{U}{\bullet}} f(e) = \sum_{v \in U} f^-(v)$$

The net flow out of U is $f^+(U) - f^-(U) = \sum_{v \in U} \left[f^+(v) - f^-(v) \right]$

Note: if f is feasible, by the conservation constraint,

$$f^+(v) - f^-(v) = \begin{cases} 0, & \text{if } v \neq s, v \neq t \\ \mathrm{val}(f), & \text{if } v = s \\ -\mathrm{val}(f), & \text{if } v = t \end{cases}$$

So for a source/sink cut $[S, T]$,

net flow out of S = $-$(net flow out of T) = $\mathrm{val}(f)$

Cor 4.3.8: If f: feasible, and $[S,T]$: source/sink cut, then $\mathrm{val}(f) \le \mathrm{cap}(S,T)$.

$$\text{Pf: } \mathrm{cap}(S,T) = \sum_{e\in[S,T]} c(e) \ge \sum_{e\in[S,T]} f(e) \ge \underbrace{f^+(S) - f^-(S)}_{\text{net flow out of } S} = \mathrm{val}(f) \qquad \square$$

cap. constraints

$$\sum_{e\in[S,T]} f(e) - \sum_{e\in[T,S]} f(e)$$

$\mathrm{val}(f)$ = net in-flow into t

= net out-flow from s