Announcements

- No class this Friday (10/27)
- No H/W this week (HW8 will be due Wed. 11/8)
- Exam 2 graded

Mean: 63.6 } out of 90
Median: 63.5 }
Std. dev.: 7.45

Problem scores:

Q_1: 93%
Q_2: 28%
Q_3: 59%
Q_4: 54%
Q_5: 93%

Menger's Theorem: If $x \neq y \in V(G)$ and $xy \notin E(G)$, then $\kappa(x,y) = \lambda(x,y)$

Pf: $\geq$) An x,y-cut must contain an internal vertex from each path in a set of pairwise internally-disjoint x,y-paths, so taking such a set of size $\lambda(x,y)$ gives $\kappa(x,y) \geq \lambda(x,y)$.

$\leq$) Induction on $n := n(G)$.

Base case: $n=2$. If $xy \notin G$, then there is no x,y-path

$\kappa(x,y) = \lambda(x,y) = 0$.

Inductive step: Let $k := \kappa(x,y) = \kappa_G(x,y)$.

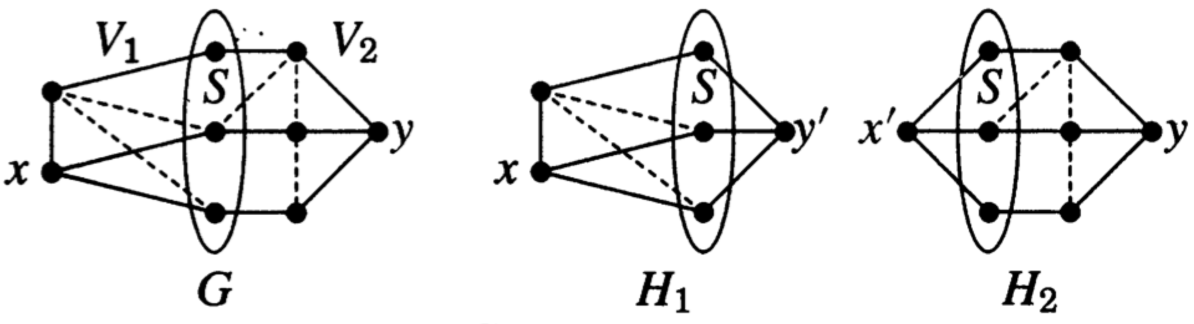
We will construct k pairwise internally-disjoint x,y-paths. Note that $N(x)$ and $N(y)$ are x,y-cuts which may or may not be minimum size.

Case I: G has a minimum x,y-cut S that isn't $N(x)$ or $N(y)$.

Let

$$V_1 = \{ v \in V(G) \mid v \text{ lies on some } \{x\}, S\text{-path} \}$$

$$V_2 = \{ v \in V(G) \mid v \text{ lies on some } S, \{y\}\text{-path} \}$$



Claim: $S = V_1 \cap V_2$

Pf of claim: Since S is a minimum x,y-cut, every vertex of S lies on an x,y-path, so $S \subseteq V_1 \cap V_2$. Conversely, if $v \in V_1 \cap V_2 \setminus S$, then following the

x,S-path from x to v followed by the S,y-path from v to y gives an x,y-path disjoint from S, a contradiction. $\square$

By similar arguments,

$$(N(x) \setminus S) \cap V_2 = \emptyset$$

$$(N(y) \setminus S) \cap V_1 = \emptyset$$

Let

$$H_1 = \underbrace{G[V_1]}_{\substack{\text{subgraph} \\ \text{induced by } V_1}} \cup \underbrace{\{y'\}}_{\substack{\text{new} \\ \text{vertex}}} \cup \underbrace{\{sy' \mid s \in S\}}_{\substack{\text{edges from every elt.} \\ \text{of } S \text{ to } y'}}$$

$$H_2 = G[V_2] \cup \{x'\} \cup \{x's \mid s \in S\}$$

Every x,y-path in G starts w/ an x,S-path (contained in H_1), so every x,y'-cut in H_1 is an x,y-cut in G, so $\kappa_{H_1}(x,y') = k$. By a similar argument, $\kappa_{H_2}(x',y) = k$.

By the inductive hypothesis, $\lambda_{H_1}(x,y') = k = \lambda_{H_2}(x',y)$, and combining these paths gives the desired internally-disjoint x,y-paths in G.

Case 2: Every minimum cut in $N(x)$ or $N(y)$.

(i) If $\exists v \in V(G) \setminus (x \cup y \cup N(x) \cup N(y))$

then v is in no minimum x,y-cut, so $\kappa_{G \setminus v}(x,y) = k$, and by the inductive hypothesis, $\exists k$ pairwise internally-disjoint x,y-paths in $G \setminus v \subseteq G$.

(ii) Otherwise, if $\exists u \in N(x) \cap N(y)$, then u appears in every x,y-cut, so $\kappa_{G \setminus u}(x,y) = k-1$. By the inductive hyp., $\exists k-1$ pairwise internally disjoint x,y-paths in $G \setminus u \subseteq G$, and all are int. dis. from the x,y-path x, u, y.

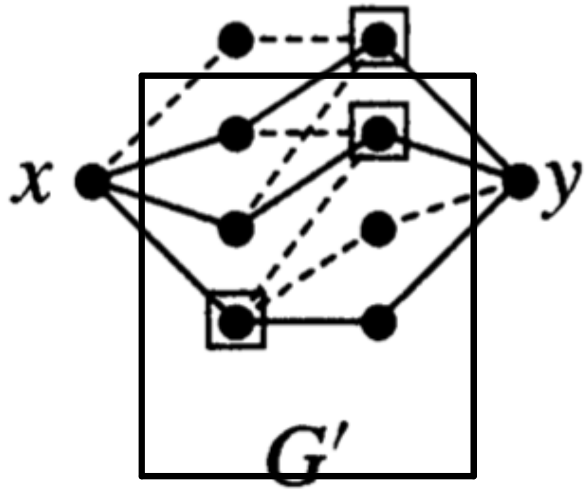
(iii) Finally, we have the case where

$$V(G) = \{x\} \sqcup \{y\} \sqcup N(x) \sqcup N(y)$$

Let $G' \subseteq G$ be the bipartite graph with

$$V(G') = N(x) \sqcup N(y)$$

$$E(G') = \{e \in E(G) \mid e \text{ has } 1 \text{ endpoint in } N(x) \text{ and the other in } N(y)\}$$



Every x,y-path uses an edge from $N(x)$ to $N(y)$, and every such edge is used in such a path, so

$$\{x,y\text{-cut in } G\} = \{\text{vertex covers in } G'\}$$

Thus, $\underbrace{\beta(G')}_{\text{min'l vertex cover size}} = k$, and by the König-Egerváry Thm., $\alpha'(G') = \beta(G')^{=k}$, so G has a matching of size k. These k edges along w/ the appropriate edges incident to x and y yield k pairwise int.-disjoint x,y-paths. $\square$

Similar results hold for directed graphs and for edge cuts

Def 4.2.11: Let D be a digraph

a) A vertex cut of D is a set $S \subseteq D$ s.t. $D \setminus S$ is not strongly connected

b) If $S, T \subseteq V(D)$, $[S,T]$ denotes the set of edges w/ tail in S and head in T. An edge cut of D is $[S, \bar{S}]$ for some nonempty $S \subseteq V(D)$.

c) (Edge)-connectivity, $\kappa(D)$, $\kappa'(D)$, $\kappa(x,y)$ defined the same w.r.t. vertex/edge cuts.

d) $\lambda(x,y)$ is still the largest # of internally-disjoint x,y-paths

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Def: Let G be a graph or digraph.

a) $\kappa'(x,y)$ = min. size of $F \subseteq E(G)$ s.t. $G \setminus F$ has no x,y-path

b) $\lambda'(x,y)$ = max. size of set of edge-disjoint x,y-paths

Thm: Let G be a graph or digraph.

a) Let $x \neq y \in V(G)$ with no edge from x to y.

   i) If G is a graph, then $\kappa(x,y) = \lambda(x,y)$ (Menger)

   ii) If G is a digraph, then $\kappa(x,y) = \lambda(x,y)$

b) Let $x \neq y \in V(G)$

   i) If G is a graph, then $\kappa'(x,y) = \lambda'(x,y)$

   ii) If G is a digraph, then $\kappa'(x,y) = \lambda'(x,y)$