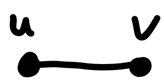
Announcement:

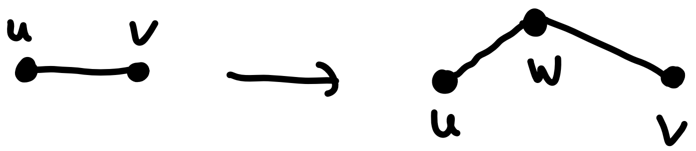
- No class this Friday (10/27)

Recall: finding conditions equivalent to 2-connectivity

Def: G: graph

a) A subdivision of an edge 

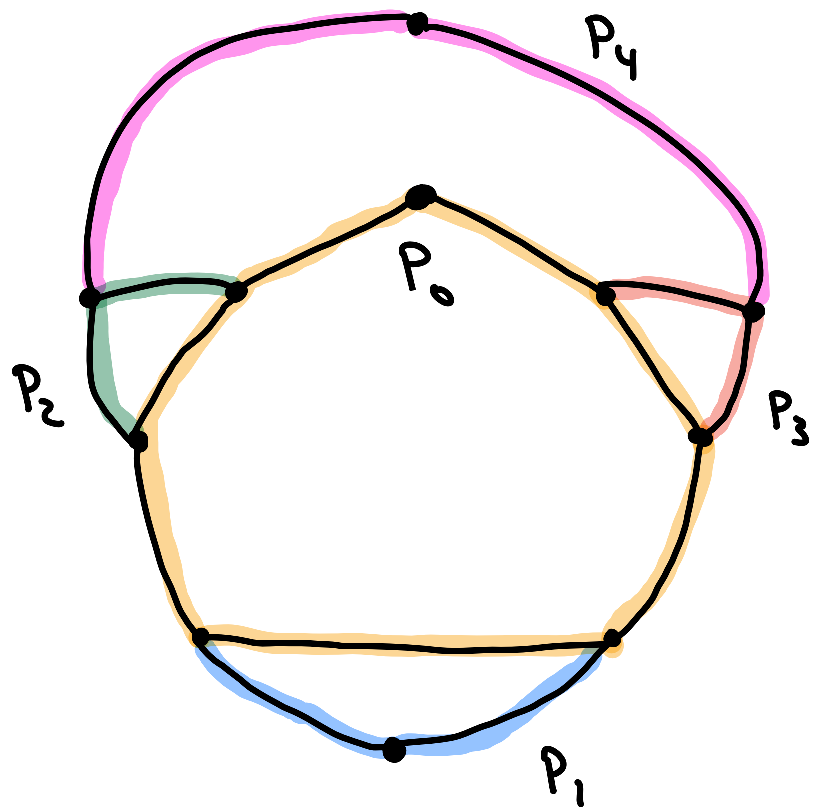
is



b) An ear of G is a max'l path whose internal vertices have degree 2 in G.

c) An <u>ear decomposition</u> of G is a decomposition $P_0, \dots, P_k$ s.t. P_0 is a cycle and For $i \geq 1$, P_i is an ear of $P_0 \cup \dots \cup P_i$

Class activity: find an ear decomposition:



Thm 4.2.8: Let $|V(G)| \geq 3$.

G is 2-connected $\Leftrightarrow$ G has an ear decomp.

Pf: $\Rightarrow$) By Thm 4.2.4 C (or D), G contains a cycle C. The ear decomp. of G is then given by the following algorithm:

Start: $G_0 = C$, $i = 0$

While $G \neq G_i$:

Let $uv \in E(G) \setminus E(G_i)$

Let $xy \in E(G_i)$

Let C' be a cycle containing uv & xy (Thm 4.2.2 D)

Let P be the maximal path in C' containing uv but no edges in G_i

Let $G_{i+1} = G_i \cup \underbrace{P_i}_{\text{ear}}$

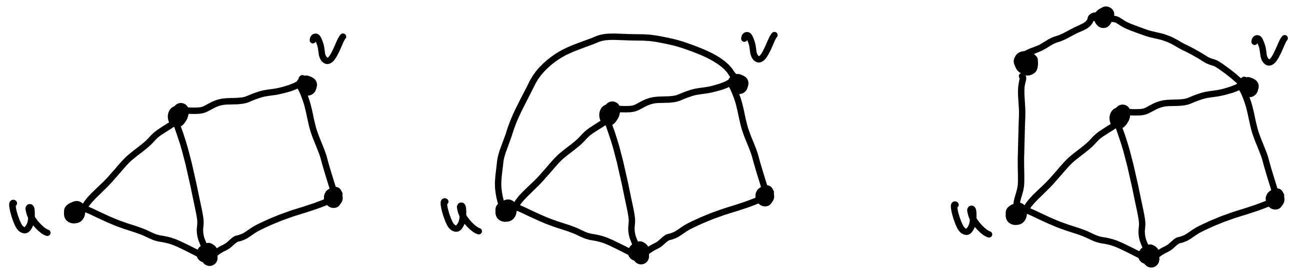
$i \leftarrow i+1$

$\Leftarrow$ We induct on the number of ears in the ear decomp.

Base case: If G is a cycle, G is 2-conn. ✓

Inductive step: Suppose G' is 2-conn. and $G = G' \cup P_k$, where P_k is an ear w/ endpoints $u, v \in V(G)$. Since G' is 2-conn., $G' \cup uv$

is 2-conn. since adding an edge never reduces $\kappa(G)$, and G is obtained by subdivisions of uv.



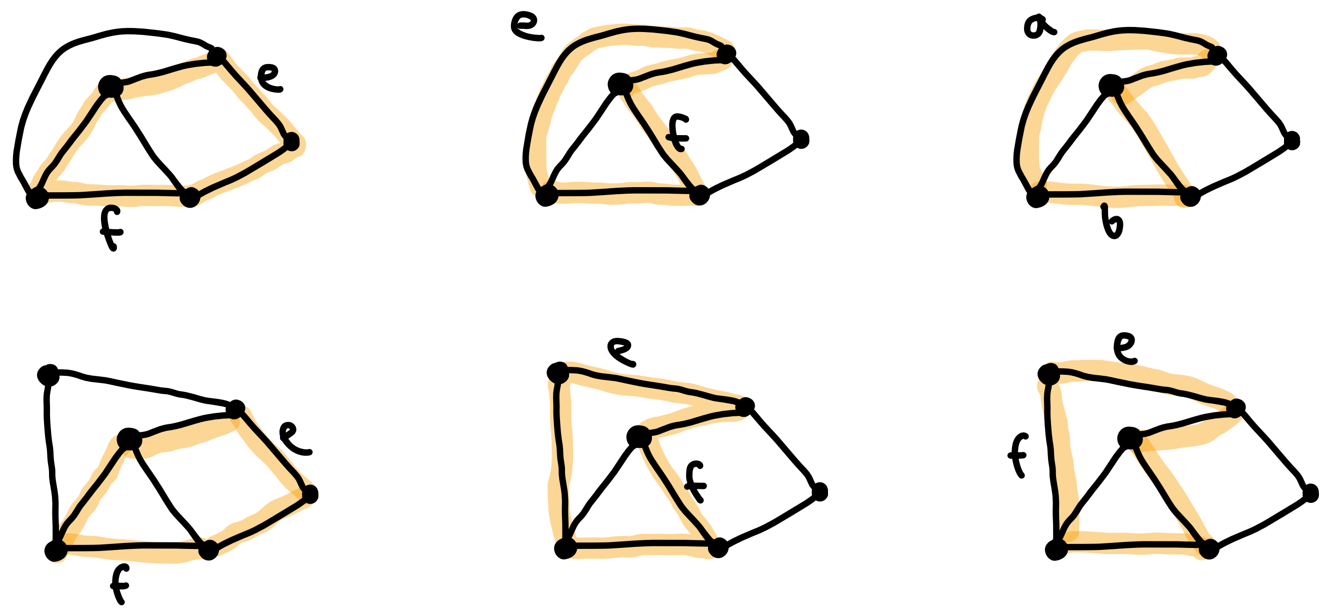
G' $\qquad$ $G' \cup uv$ $\qquad$ G

Thus, the result follows from:

Claim (4.2.6): If H is 2-conn and H' is obtained from H by subdividing an edge, then H' is 2-conn.

Proof by picture:

(using 4.2.4D: H' 2-conn. $\Leftrightarrow$ $\delta(H') \geq 1$ and every pair of edges lie on a common cycle)



$\square$

Cor: Let G be a graph w/ ≥ 3 vertices. TFAE:

A) G is conn. and has no cut-vertex

B) $\forall x,y \in V(G)$, $\exists$ internally-disjoint x,y-paths

C) $\forall x,y \in V(G)$, $\exists$ cycle containing x and y

D) $\delta(G) \geq 1$, and $\forall e,f \in E(G)$, $\exists$ cycle containing e and f

E) G is 2-conn.

F) G has an ear decomposition

Can generalize part of this to k-conn. graphs

Def 4.2.15:

a) If $X, Y \subseteq V(G)$, an X,Y-path is a path w/ first vertex in X, last vertex in Y, and no other vertices in $X \cup Y$.

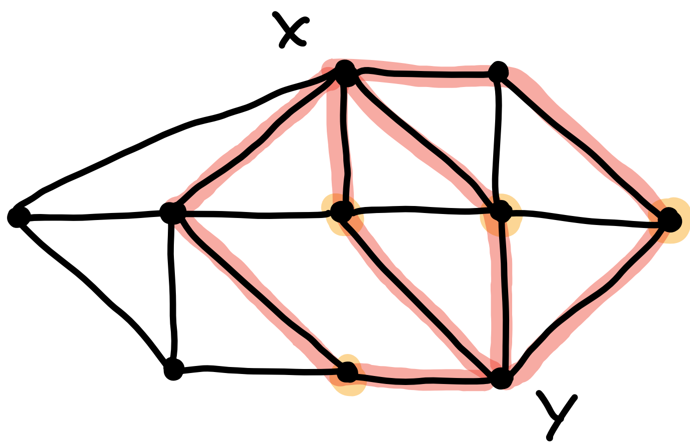
b) $S \subseteq V(G)$ is an x,y-cut if $G \setminus S$ has no x,y-path
$\in V(G)$

c) $\kappa(x,y)$ is the minimum size of an x,y-cut

$$\left[\text{i.e. } \kappa(G) = \min_{x,y \in V(G)} \kappa(x,y)\right]$$

d) $\lambda(x,y)$ is the maximum size of a set of pairwise internally disjoint x,y-paths

Class activity: Compute $\kappa(x,y)$ and $\lambda(x,y)$



$\kappa(x,y) = 4$

$\lambda(x,y) = 4$

Menger's Theorem: If $x \neq y \in V(G)$ and $xy \notin E(G)$, then $\kappa(x,y) = \lambda(x,y)$

Pf: $\geq$) An x,y-cut must contain an internal vertex from each path in a set of pairwise $\underbrace{\text{internally-disjoint } x,y\text{-paths}}_{}$, so taking a set of size $\lambda(x,y)$ gives $\kappa(x,y) \geq \lambda(x,y)$.

$\leq$) Next time