Announcement:

HW7 posted (due Wed. 10/25)

Recall:

$\kappa(G)$ = min. size of $S \subseteq V(G)$ s.t. $G \setminus S$ is disconn.

$\kappa(G)$ = min. size of $F \subseteq E(G)$ s.t. $G \setminus F$ is disconn.

Whitney's Thm: $\kappa(G) \le \kappa'(G) \le \delta(G)$ if G: simple

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Today's goal: give several characterizations of 2-connected graphs.

Def 4.2.1: Two u,v-paths are <u>internally disjoint</u> if their intersection is $\{u, v\}$.

Thm 4.2.2: Let G be a graph w/ ≥ 3 vertices. Then,

G is 2-connected $\iff$ $\forall u, v \in V(G)$, $\exists$ two internally disjoint u,v-paths

Pf: $\Leftarrow$) If G has internally disjoint u,v-paths for all $u,v \in V(G)$, then deleting one vertex from G leaves ≥ 1 u,v-path for all remaining vertices u and v.

$\Rightarrow$) Suppose that G is 2-conn. and let $u,v \in V(G)$.

Use induction on $d(u,v)$

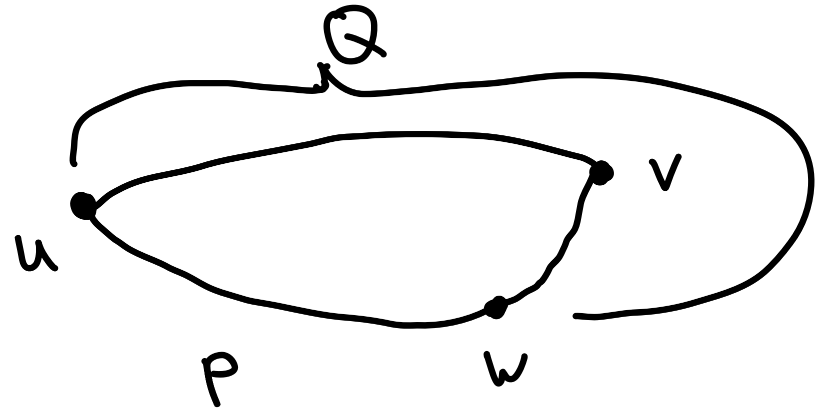
Base case: $d(u,v) = 1$. Since by Whitney's Thm.

$\kappa'(G) \geq \kappa(G) \geq 2$, $G \setminus \underbrace{uv}_{\text{any edge}}$ is conn.

Thus, $\exists$ a u,v-path in $G \setminus uv$, and this is internally disjoint from the path u, uv, v.

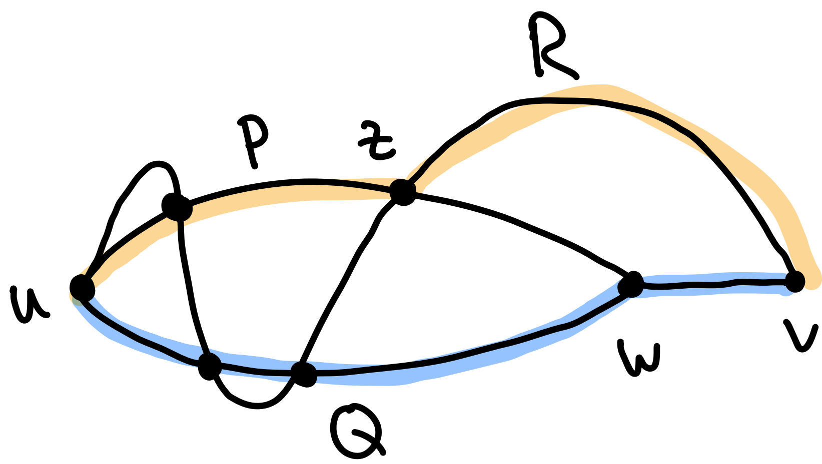
Inductive step: Let $k := d(u,v) \geq 2$. Choose any minimum-length u,v-path, and let w be the vertex next to v on this path, so $d(u,w) = k-1$.

By the inductive hyp., G has internally-disjoint u,w-paths P and Q. If $v \in V(P) \cup V(Q)$, then we obtain internally disjoint u,v-paths



Now assume $v \notin V(P) \cup V(Q)$. Since G is 2-conn., $G \setminus w$ is conn. and thus contains a u,v-path R.

Let z be the last vertex before v belonging to $P \cup Q$ (WLOG, say $z \in P$)



Since P and Q are internally disjoint, the u,z-subpath of P followed by the z,v-subpath of R is internally disjoint from Q followed by wv. $\square$

Thm 4.2.2: Let G be a graph w/ ≥ 3 vertices. TFAE:

the following are equiv.

A) G is conn. and has no cut-vertex

B) $\forall x, y \in V(G)$, $\exists$ internally-disjoint x,y-paths

C) $\forall x, y \in V(G)$, $\exists$ cycle containing x and y

D) $\delta(G) \geq 1$, and $\forall e, f \in E(G)$, $\exists$ cycle containing e and f

E) G is 2-conn.

Pf: $A \Leftrightarrow E$: Def'n of 2-conn.

$B \Leftrightarrow E$: Thm. 4.2.2.

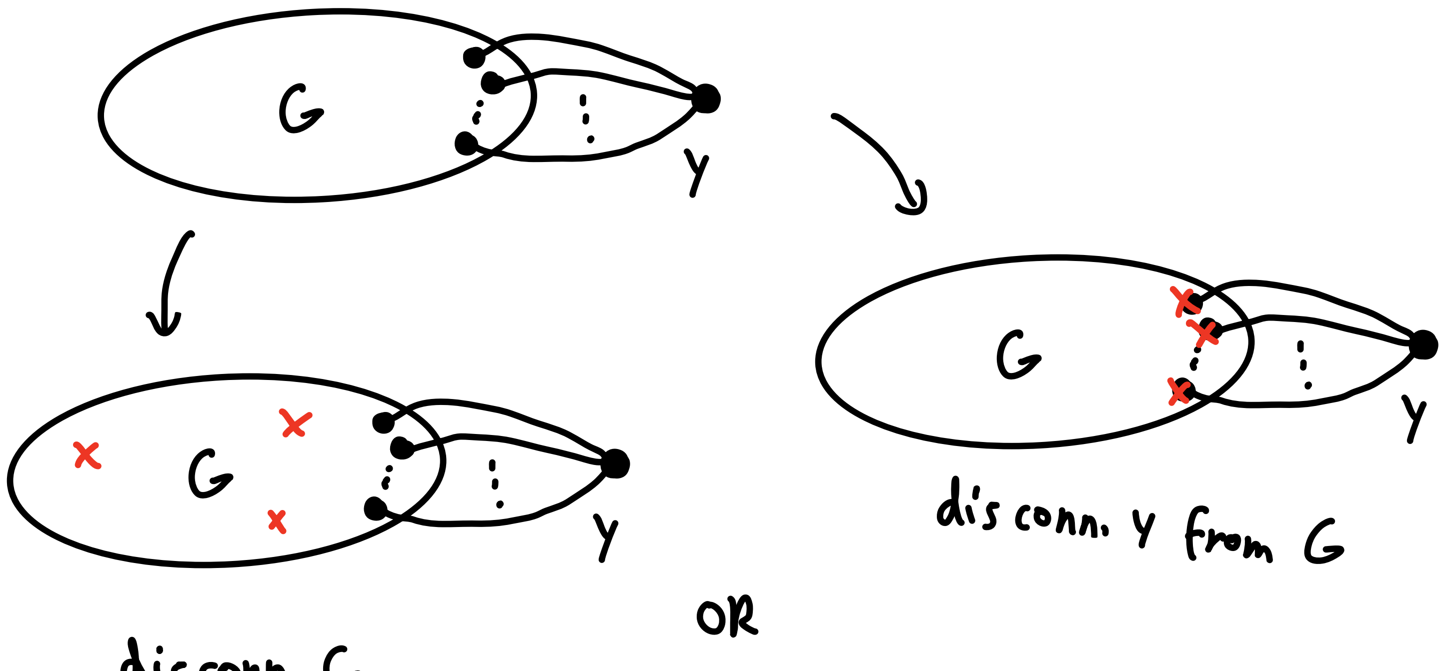
$B \Leftrightarrow C$: $\left\{ \begin{array}{c} \text{cycles containing} \\ u \text{ and } v \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{pairs of internally-} \\ \text{disjoint } u,v\text{-paths} \end{array} \right\}$

$D \Rightarrow C$: $\delta(G) \geq 1 \Rightarrow x$ and y are not iso. If e is incident to x and f is incident to y, by D, $\exists$ cycle containing e and f, so containing x and y. (If e is incident to <u>both</u> x and y, let f be any other edge).

Interlude:

Expansion Lemma (4.2.3): If G is k-conn. and G' is obtained from G by adding a new vertex y w/ $\geq k$ neighbors in G, then G' is k-conn.

Pf by picture:



$\square$

Finish pf of Thm 4.2.4:

$A, C, E \Rightarrow D$: Since G is conn., $\delta(G) \geq 1$.

Let $uv, xy \in E(G)$. Let G' be the graph w/

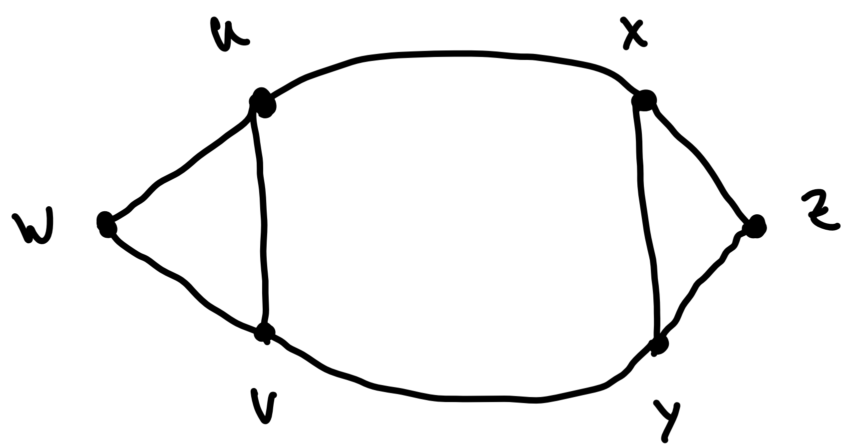
$\notin G$

$$V(G') = V(G) \cup \{w, z\}$$

$$E(G') = E(G) \cup \{uw, vw, xz, yz\}$$

By the expansion lemma, G' is 2-conn. so since $E \Leftrightarrow C$, w and z lie on a cycle C in G'.

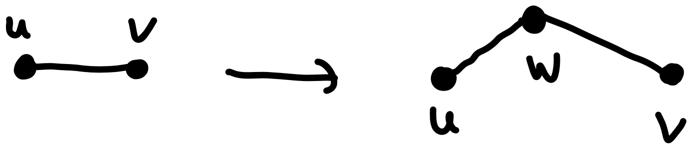
Since $d(w) = d(z) = 2$, C contains the paths u, w, v and x, z, y, but not the edges uv and xy. Replacing u, w, v with uv and x, z, y w/ xy gives the desired cycle in G. $\square$



Def : G : graph

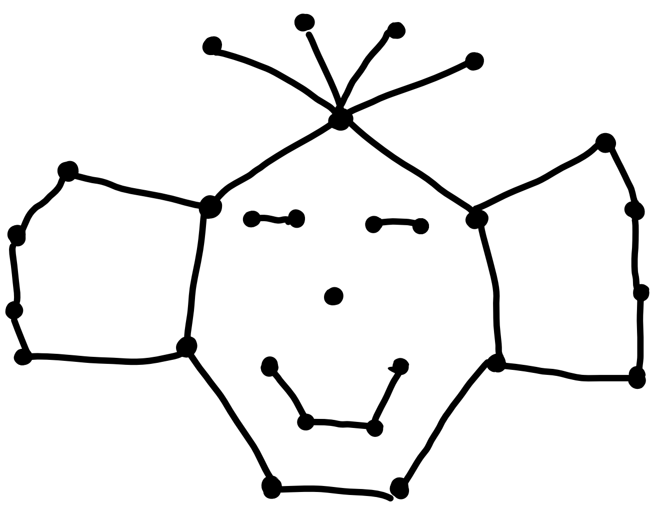
a) A subdivision of an edge 

is



b) An ear of G is a max'l path whose internal vertices have degree 2 in G.

Class activity: Find the ears!



c) An <u>ear decomposition</u> of G is a decomposition $P_0, \ldots, P_k$ s.t. P_0 is a cycle and for $i \geq 1$, P_i is an ear of $P_0 \cup \ldots \cup P_i$

Class activity: find an ear decomposition: