Announcement:

Midterm 2 tonight!

7:00pm - 8:30pm in 217 Noyes Lab. (ref. sheet allowed)

Be early!

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Exam covers: Ch 1-3 (focus on Ch. 2, 3),
plus circuit application of matrix tree thm.

Most focus: topics that appeared in lecture or homework
Some focus: topics in relevant subsections of textbook
Low/no focus: topics in subsections we didn't cover at all

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Types of graphs: (dis.)conn., bipartite, paths, cycles, trees, forests, complete (bipartite) graphs, digraphs, weighted graphs

Walks, trails, circuits

Things graphs have:

Eulerian circuits (Euler Thm.)

Perfect matching (Hall's Thm., Tutte's Thm.)

Trees:

- Equiv. def'ns
- Prüfer code & Cayley's formula
- Spanning subgraphs & spanning trees
- Matrix tree thm.
- Kirchoff's Laws and Kirchoff's Thm.

Algorithms:

- Kruskal (min. wt. spanning tree)
- Dijkstra (distances)
- Gale-Shapley (stable matching)
- Algorithmic thinking

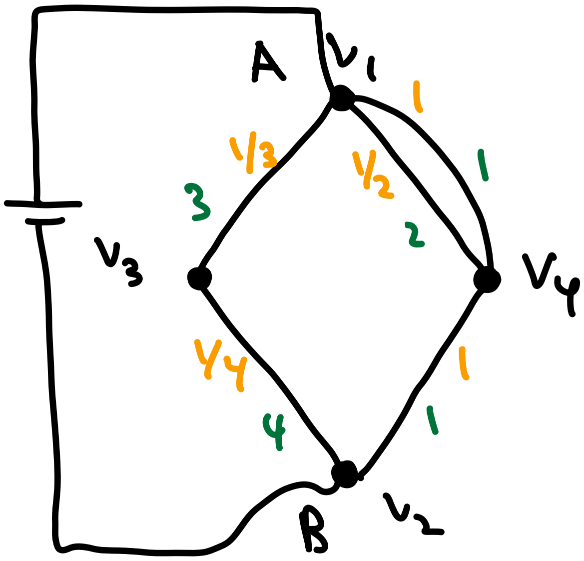
Matchings:

- Perfect vs. $\overbrace{\text{maximum vs. maximal}}^{\text{general concept}}$
- M-alt. paths & M-aug. paths
- Theorems: Berge, Hall, Tutte, Berge-Tutte, Petersen x2
- Relationships btwn. matchings, vertex/edge covers, and indep. sets
- k-factors

Examples:

1) Consider the graph $G$ w/ resistances in yellow

conductances in green




Find the effective resistance from A to B

Sol'n: Step 1: Compute $\tau(G)$ using conductances as the weights

Method 1: Matrix tree thm.

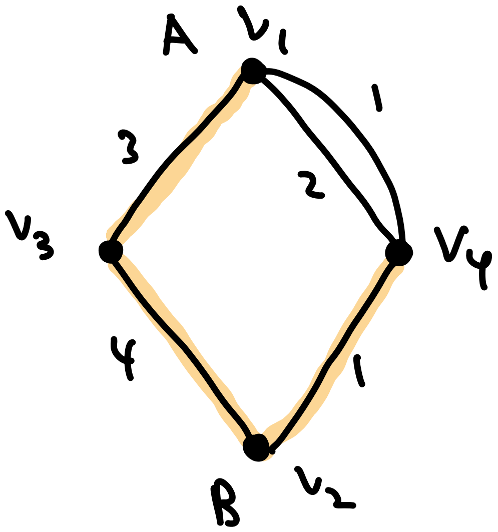
$$L(G) = \begin{bmatrix} 6 & 0 & -3 & -3 \\ 0 & 5 & -4 & -1 \\ -3 & -4 & 7 & 0 \\ -3 & -1 & 0 & 4 \end{bmatrix} \qquad L^3(G) = \begin{bmatrix} 6 & 0 & -3 \\ 0 & 5 & -1 \\ -3 & -1 & 4 \end{bmatrix}$$

By the MTT,

$$\tau(G) = \det L^3(G) = 6 \det \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix} - 0 \det \begin{bmatrix} 0 & -1 \\ -3 & 4 \end{bmatrix}$$

$$+ (-3) \det \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} = 6 \cdot 19 - 3 \cdot 15 = 69$$

OR Method 2 : directly

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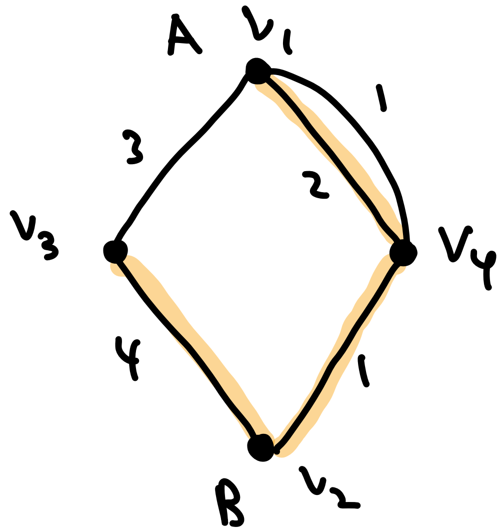


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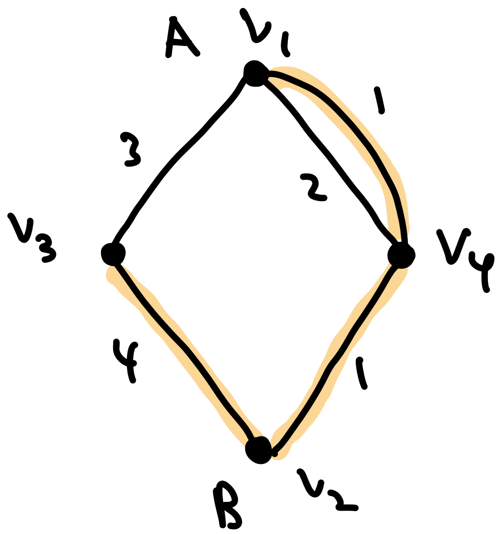


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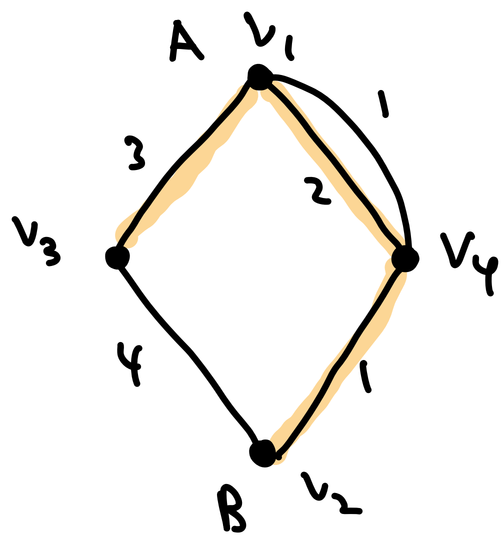


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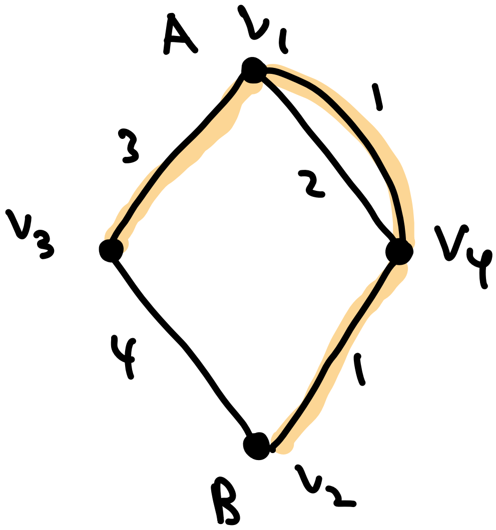


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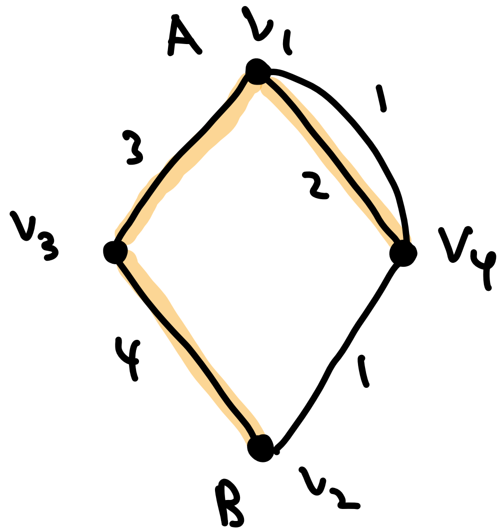


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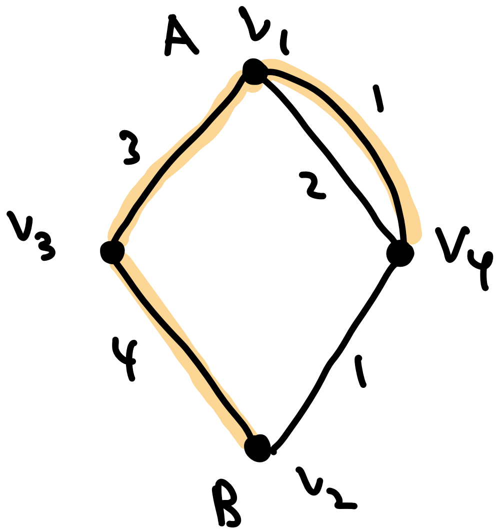


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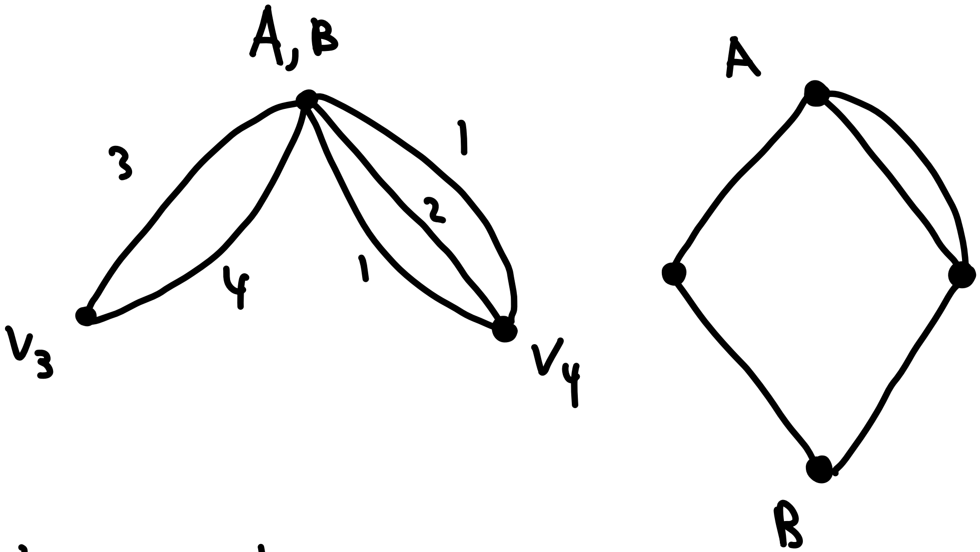
$$T(G) = 12+8+4+6+3+24+12 = 69$$

Step 2:

$(G \sqcup AB)\cdot AB$ :




$\tau(\quad) = 28$ by similar methods

Step 3: By Kirchoff's Thm., effective resistance of G is

$$\frac{\tau((G \sqcup AB)\cdot AB)}{\tau(G)} = \frac{28}{69}$$

$\boxed{\text{Fixed}}$

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2) Let G be a simple graph s.t. $\delta(G) \geq k$ and $n(G) \geq 2k$. Prove that G has a matching of size $\geq k$.

Pf: $k=0$: Clear So assume $k \geq 1$.

By the Berge-Tutte formula, it suffices to show that $\forall S \subseteq V(G)$,

$$o(G \setminus S) - |S| \leq n - 2k$$

Suppose that $S \subseteq V(G)$ and

$o(G\setminus S) - |S| > n-2k$. Let $s = |S|$, so

$o(G\setminus S) \underset{(*)}{>} n-2k+s$ and since $n \geq o(G\setminus S) + |S|$,

$n-s > n-2k+s$, so $n > n-2k+2s$, so $k > s$.

This means that all $v \in V(G)\setminus S$ have $< k$ neighbors in $S$, so since $\delta(G) \geq k$, $v$ must have at least $k-s$ other neighbors, and so each component of $G\setminus S$ has $\underset{(**)}{\geq} 1+k-s$ vertices,

so

$$(1+k-s)(n-2k+s+1) + s \leq n$$

$$(k-s)(n-2k+s+1) + n-2k+2s+1 \leq n \quad \text{(mult. out left side)}$$

$$(k-s)(n-2k+s+1) + 2(s-k) < 0 \quad \text{(subtract } n \text{ from both sides)}$$

$$\underbrace{(k-s)}_{\substack{>0 \text{ since} \\ k \geq s}}\underbrace{(n-2k+s-1)}_{\substack{\geq 0 \text{ unless} \\ s=0 \text{ i.e. } S=\emptyset}} < 0 \qquad \text{(combine terms)}$$

So this is a contradiction unless $S = \emptyset$.

$S = \emptyset$ case: need to show $o(G) \leq n-2k$

Since $\delta(G) \geq k$, each component of $G$ has $\geq k+1$ vertices, so

$$(k+1)\, o(G) \leq n$$

$$o(G) \leq n - o(G)k,$$

and we're done if $o(G) \geq 2$.

Finally, if $o(G) = 1$, then $n$ is odd, so since $n \geq 2k$, $n-2k \geq 1$, so

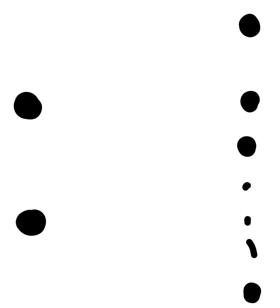
$$o(G) = 1 \leq n-2k. \qquad \square$$

3) Compute $\tau(K_{2,m})$

Sol'n 1:

Let $X, Y$ be the partite sets, w/ $|X| = 2$, $|Y| = m$. Let $T$ be a spanning tree of $K_{2,m}$.

Since $T$ is conn.,



$\exists!$ vertex $v \in Y$ adjacent to both vertices in $X$ ($m$ choices). Every other vertex is adj. in $T$ to exactly one vertex in $X$ (2 choices per vertex in $Y \setminus \{v\}$; $2^{m-1}$ choices in total), and any such set of choices produces a (distinct) spanning tree. Thus, $\exists\, m \cdot 2^{m-1}$ spanning trees of $K_{2,m}$.

Sol'n 2:

$$L(K_{2,m}) = \begin{bmatrix} m & 0 & -1 & \cdots & -1 \\ 0 & m & -1 & \cdots & -1 \\ -1 & -1 & 2 & & 0 \\ \vdots & \vdots & & \ddots & \\ -1 & -1 & 0 & & 2 \end{bmatrix}$$

$$L'(K_{2,m}) = \begin{bmatrix} m & -1 & \cdots & -1 \\ -1 & 2 & & 0 \\ \vdots & & \ddots & \\ -1 & 0 & & 2 \end{bmatrix}$$

Row operations: add every other row to top row

$$\det L'(K_{2,m}) = \det \begin{bmatrix} 0 & 1 & \cdots & 1 \\ -1 & 2 & & 0 \\ \vdots & & \ddots & \\ -1 & 0 & & 2 \end{bmatrix}$$

Col operations: add $\frac{1}{2}$ of every col. to first col.

$$\det L'(K_{2,m}) = \det \begin{bmatrix} m/2 & 1 & \cdots & 1 \\ 0 & 2 & & 0 \\ \vdots & & \ddots & \\ 0 & & 0 & 2 \end{bmatrix} = m \cdot 2^{m-1}$$