Announcements:

- Quiz today!
- Midterm 2 next Wed.

  Wed. 10/18 7:00pm - 8:30pm in 217 Noyes Lab.

  See email for policies

Recall: Tutte's Thm.

$o(G) :=$ # odd order components of G

G has a perfect matching $\iff o(G \setminus S) \leq |S| \; \forall S \subseteq V(G)$

Cor 3.3.7 [Berge-Tutte Formula]:

The number of vertices u unsaturated by a maximum matching of G is

$$d := \max_{S \subseteq V(G)} \{ o(G \setminus S) - |S| \}$$

Pf: For any $S \subseteq V(G)$, at most $|S|$ edges can match vertices of S to vertices in odd components of $G \setminus S$. Any extra odd components will have a vertex left over, so every matching has $\geq o(G \setminus S) - |S|$ unsaturated vertices, and so

$$u \geq \max_{S \subseteq V(G)} \{ o(G \setminus S) - |S| \} = d$$

We know $d \geq 0$ since $o(G \setminus \emptyset) - |\emptyset| \geq 0$.

Define G' as:

$$V(G') = V(G) \cup V(K_d)$$

"join of G and K_d"

$$E(G') = E(G) \cup E(K_d) \cup \{ uv \mid u \in V(G), v \in V(K_d) \}$$

If G' has a perfect matching, then G has a matching w/ $\leq d$ unsaturated vertices, since deleting the d added vertices eliminates edges that saturate at most d vertices of G, so we'll have $u \leq d$.

For any S,
$$n(G \setminus S) \equiv o(G \setminus S) \pmod{2}$$
$$n(G) - |S| \equiv o(G \setminus S) \pmod{2}$$

$$n(G) \equiv o(G \setminus S) \pmod{2}$$
$$n(G) \equiv d \pmod{2}$$

So $n(G') = n(G) + d$ is even

Evaluate Tutte's condition on G':

Let $S' \subseteq V(G')$. WTS: $o(G' \setminus S') \leq |S'|$

a) $S' = \emptyset$ ✓

b) $S' \neq \emptyset$ but $V(K_d) \nsubseteq S'$: $G' \setminus S'$ has 1 component, so $o(G' \setminus S') \leq 1 \leq |S'|$

c) $V(K_d) \subseteq S'$: Let $S = S' \setminus V(K_d)$.

Then $G' \setminus S' = G \setminus S$, so

$$o(G' \setminus S') = o(G \setminus S) \underset{\substack{\uparrow \\ \text{by def'n} \\ \text{of } d}}{\leq} |S| + d = |S'| \qquad \square$$

Cor 3.3.8 [Petersen, 1891]: Every 3-regular graph w/ no cut-edge has a perfect matching

Pf: Let G be a 3-reg graph w/ no cut edge. We prove that G satisfies Tutte's condition. Let $S \subseteq V(G)$; we count edges btwn. S and odd components of $G \setminus S$. Since G is 3-reg., each vertex of S is incident to ≤ 3 such vertices. If each odd component H of $G \setminus S$ is incident to

≥ 3 of these edges, then $o(G \setminus S) \leq |S|$.
Let m be # edges from S to H. WTS: $m \geq 3$.
The deg. sum of H is $G \setminus S$ is
$3n(H) - m$, so this is even, and since
$n(H)$ is odd, m is odd. Since G has no
cut edge, $m \neq 1$, so $m \geq 3$ $\square$

Def 3.3.1: A k-factor is a spanning
k-regular subgraph

Special case: perfect matching $\equiv$ 1-factor

Consequence of Hall's Thm.:

Cor 3.1.13: If $k > 0$, every k-regular
bipartite graph has a perfect matching

Thm 3.3.9 [Petersen, 1891]: Every regular graph of even pos. degree has a 2-factor

Pf: Let G be $2k$-regular w/ $V(G) = \{v_1, \ldots, v_n\}$.
If G is conn., it has an Eulerian circuit C by Thm 1.2.26. Let H be a U,W-bigraph w/ $U = \{u_1, \ldots, u_n\}$, $W = \{w_1, \ldots, w_n\}$ and $u_i \bullet\!\!-\!\!\!-\!\!\bullet\, w_j$ iff C traverses an edge from v_i to v_j.
Then, H is k-regular, so by Cor. 3.1.13, it has a perfect matching M. Create the spanning subgraph $N \subseteq G$ where for each edge $u_i \bullet\!\!-\!\!\!-\!\!\bullet\, w_j$ in M, $v_i \bullet\!\!-\!\!\!-\!\!\bullet\, v_j$ is in N.

Therefore,

$$\deg_N(v_i) = \deg_M(u_i) + \deg_M(w_i)$$
$$= 1 + 1 = 2$$

So N is a 2-factor of G. If G is not conn., apply the above to its conn. components. ☐

A related idea allowed Tutte to find a necessary and sufficient condition for G to have a k-factor for any k, or, even more generally, a subgraph w/ <u>any</u> degree sequence (see optional subsection)