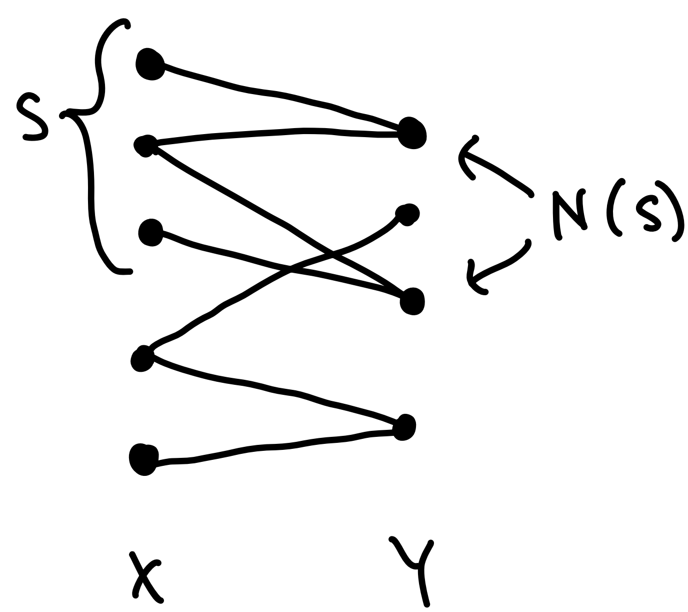
Hall's (Marriage) Thm (3.1.11): Let G be an X,Y-bigraph. Then,

G has a matching that saturates X $\iff$ $|N(S)| \geq |S|$ for all $S \subseteq X$

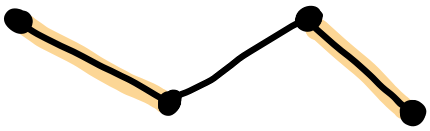
Pf: $\Rightarrow$ If G has such a matching M, the vertices in S are matched to $|S|$ vertices, all of which must be in $|N(S)|$



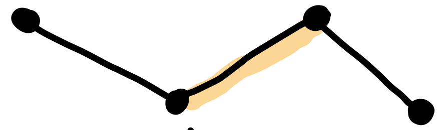
$\Leftarrow$ Need a def'n first

Def 3.1.6: Let $M \subseteq G$ be a matching.

a) An M-alternating path is a path $P \subseteq G$ which alternates btwn. edges in M and edges not in M

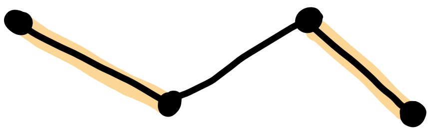


M-alt.

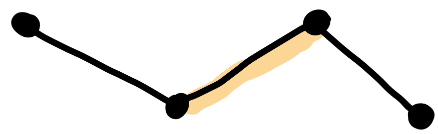


M-alt.

b) An M-augmenting path is an M-alternating path whose endpoints are unsaturated

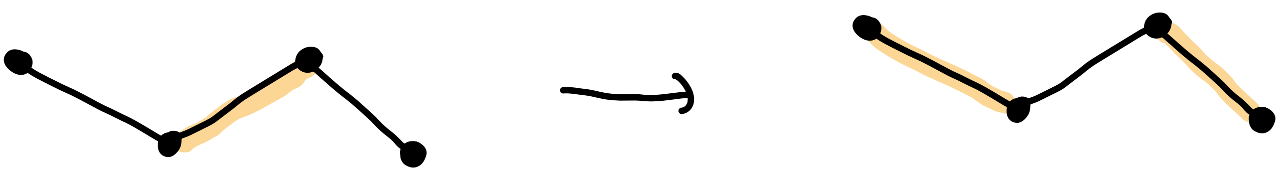


not M-aug.



M-aug.

Idea: given an M-augmenting path, swap the edges and non-edges



Always gives a larger matching

Thm 3.1.10: Let $M \subseteq G$ be a matching. Then, M is maximum $\Leftrightarrow$ G has no M-augmenting path

Pf: We prove the contrapositive.

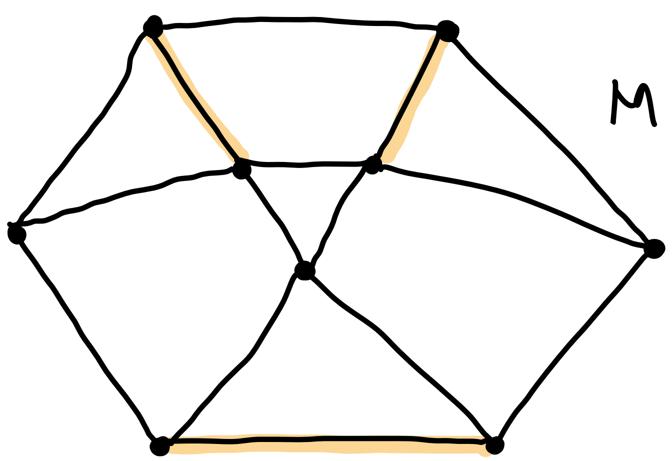
$\Rightarrow$ If G has an M-aug. path P, the process above shows that M is not maximum.

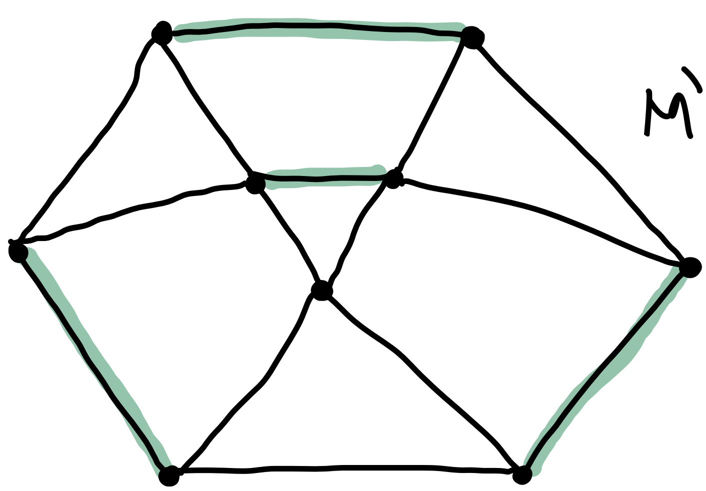
$\Leftarrow$ If M is not maximum, let M' be a larger matching in G. Let F be following graph:

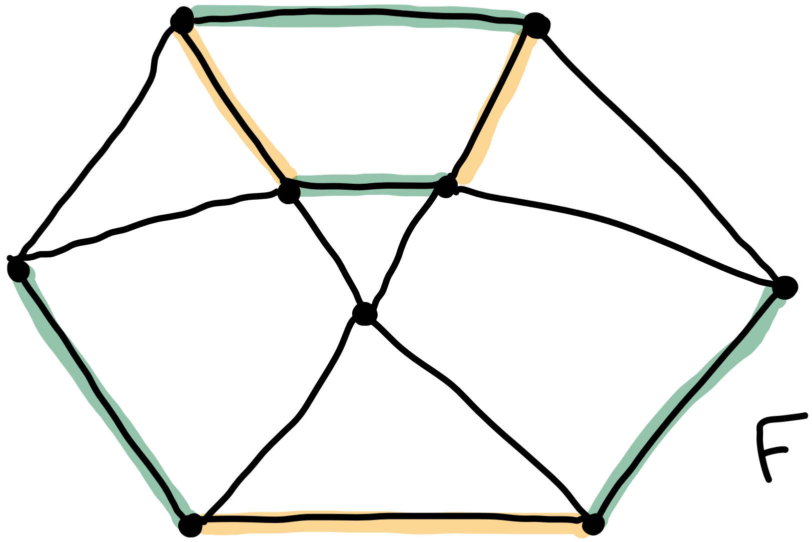
$V(F) = V(G)$

$E(F) = \{e(G) \mid e \in \text{exactly one of } M, M'\}$

"symmetric difference"







Edges in F must alternate btwn. edges in M and M' along any trail. Thus, the deg. in F of any vertex is ≤ 2, and F is made up of even cycles and paths. Each cycle must have an equal num. of edges from M and M', so since $|E(M)| < |E(M')|$, at least one path in F must have one more edge from M' than M, and this is an M-aug. path. $\square$

Back to pf of Hall's Thm:

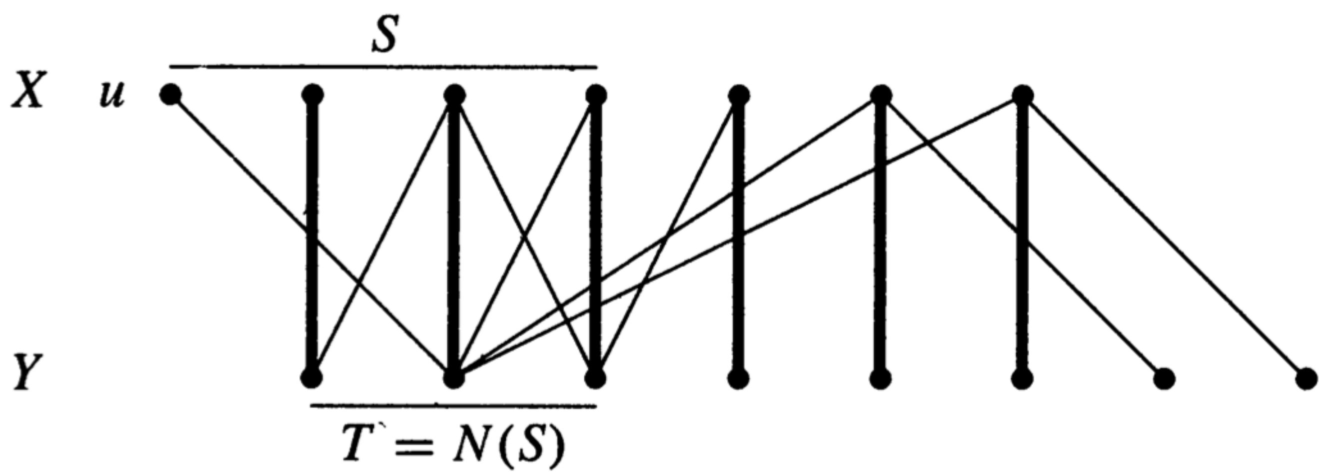
G has a matching that saturates X $\iff$ $|N(S)| \geq |S|$ for all $S \subseteq X$

$\Leftarrow$ Prove contrapositive: if M is a maximum matching that doesn't saturate X, then $\exists S \subseteq X$ s.t. $|N(S)| < |S|$.

Let $u \in X$ be unsaturated by M, and let:

$S = \{v \in X \mid \exists\, M\text{-alt. } u,v\text{-path}\}$

$T = \{v \in Y \mid \exists\, M\text{-alt. } u,v\text{-path}\}$



We have $v \in S \setminus \{u\}$ iff M matches v w/ a vertex in T. Furthermore every vertex of T is saturated; otherwise G would contain an M-augmenting path, which contradicts Thm. 3.1.10. Thus, $|T| = |S \setminus \{u\}| = |S| - 1$.

Now consider $N(S)$. If $y \in Y \setminus T$ has a neighbor $v \in S$, then $vy \notin M$ since every edge in M incident to S has its other endpoint in T. But since $vy \notin M$, adding vy to an M-alt. u,v-path gives an M-aug path, contradicting the assumption that M is maximum. Thus, $N(S) \subseteq T$, $|N(S)| \leq |T| = |S| - 1 < |S|$ $\square$

Def (3.1.14/3.1.19): Let G be a graph

a) $Q \subseteq V(G)$ is a <u>vertex cover</u> of G if every edge in $E(G)$ has ≥ 1 endpoint in Q

b) $L \subseteq E(G)$ is an <u>edge cover</u> of G if every vertex in $V(G)$ is incident to ≥ 1 edge in L

c) $\alpha(G) :=$ maximum size of independent set
$\alpha'(G) :=$ maximum size of matching
$\beta(G) :=$ minimum size of vertex cover
$\beta'(G) :=$ minimum size of edge cover