Announcement:

HW6 posted (due next Wed. 10/11)

Recall: Dijkstra's Algorithm

Input: A weighted graph G and a vertex $u \in V(G)$

Start: $S = \{u\}$, $t(u) = 0$,

$$t(z) = \min_{u \overset{e}{\bullet\!\!-\!\!\bullet} z} wt(e) \quad \text{if } z \neq u$$

While $\exists z \notin S,\ t(z) < \infty$:

Choose $v \notin S$ s.t. $t(v) = \min_{z \notin S} t(z)$

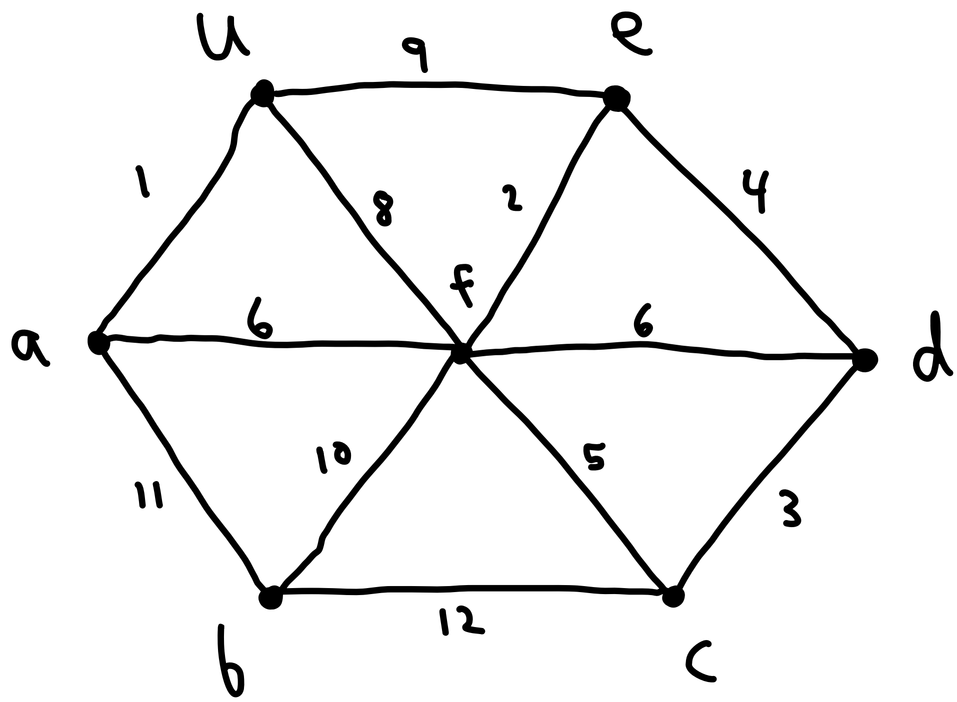
Add v to S

For all edges $v \overset{e}{\bullet\!\!-\!\!\bullet} z$, $z \notin S$:

Replace $t(z)$ w/ $\min(t(z), t(v) + wt(e))$

Output: $t(v) = d(u,v)$ for all $v \in V(G)$

Class activity: Dijkstra!



S

u a f

e b c d

$t(u) = 0$

$t(a) = 1$

$t(b) = 12$

$t(c) = 12$

$t(d) = 13$

$t(e) = 9$

$t(f) = 7$

Thm 2.3.7: The output of Dijkstra's Algorithm $t(v)$ is always the distance function $d(u,v)$.

Pf: We prove the stronger statement that after each iteration:

1) For $z \in S$, $t(z) = d(u,z)$

2) For $z \notin S$, $t(z)$ is the least length of a u,z-path reaching z directly from S.

Induction of $k := |S|$.

Base case: $k = 1$

1) $S = \{u\}$ and $t(u) = 0 = d(u,u)$

2) If $z \neq u$, $t(z) = \min_{e = uz} wt(e)$, and the only paths reaching z directly from S are edges.

Inductive step: Suppose $|S| = k$ and 1) & 2) hold

Choose $v \notin S$ s.t. $t(v) = \min_{z \notin S} t(z)$, and let

$S' = S \cup v$. We prove 1) and 2) for S'.

1) By the inductive hyp., this holds for all elts. of S' except v.

Since $v \notin S$, if P is a u,v-path w/ $wt(P) = d(u,v)$, let w be the second last vertex in P.

If $w \notin S$, by the inductive hyp., $t(v)$ (resp. $t(w)$) is the least length of a u,v-path (resp. u,w-path) reaching v (resp. w) directly from S.

By assumption, $t(v) \le t(w)$, so

$$t(w) \le wt(P) = d(u,v) \le t(v) \le t(w),$$

and so we have equality.

If $w \in S$, by the inductive hyp., $t(v) = wt(P) = d(u,v)$.

2) Let $z \notin S'$. Let P be a min-length u,z-path reaching z directly from S'. If the second-last vertex in P is in S, then by the inductive hyp., $wt(P)$ is the previous value of $t(z)$; call this $t'(z)$. Otherwise, the second-last vertex of P is v, and then

$$wt(P) = d(u,v) + wt(vz) = t(v) + wt(vz).$$

In either case, we have

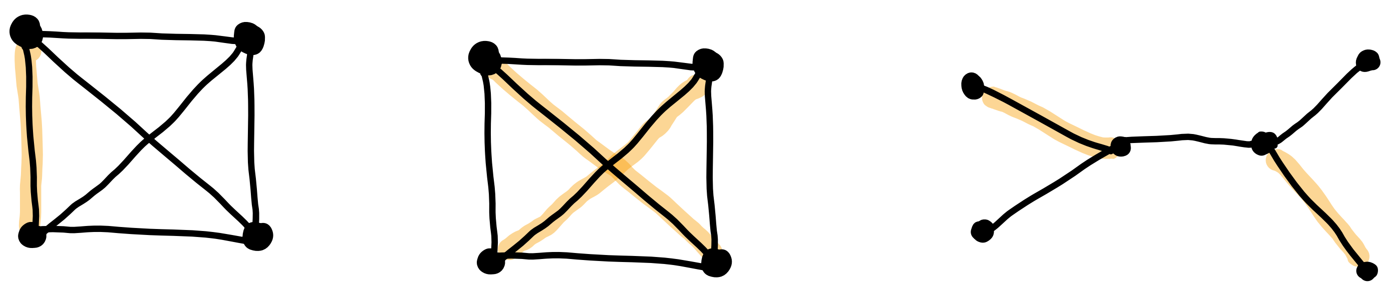
$$wt(P) = \min\left(t'(z), t(v) + wt(v,z)\right) = t(z). \qquad \square$$

Special case: breadth-first search (all weights are 1)

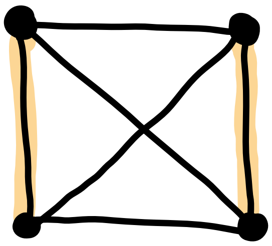
Chapter 3: Matchings and Factors

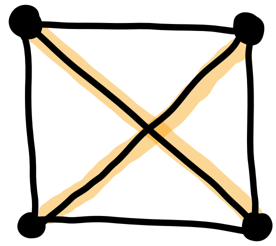
Def 3.1.1/3.1.4: Let G be a graph

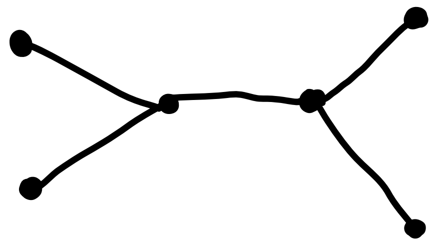
a) A <u>matching</u> in G is a spanning subgraph $M \subseteq G$ such that each vertex has degree ≤ 1 in M



b) A <u>perfect matching</u> is a matching $M \subseteq G$ such that each vertex has degree exactly 1 in M



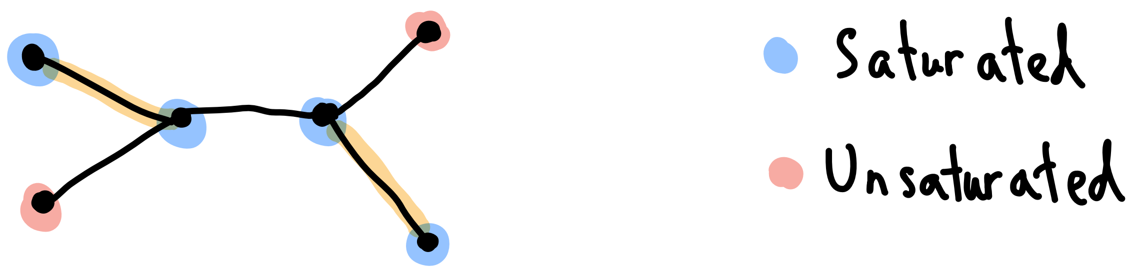




none exists

c) We call a vertex saturated if it has deg. 1 in M

We call a vertex unsaturated if it has deg. 0 in M

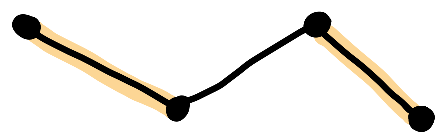


d) M is a maximal matching if there is no matching M' with $M \subsetneq M' \subseteq G$

M is a maximum matching if there is no matching M' with $|E(M)| < |E(M')|$

Class activity: Maximal? Maximum? Perfect?

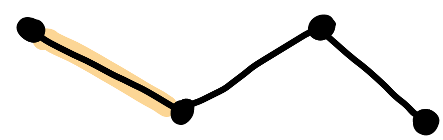
a)



maximal
maximum

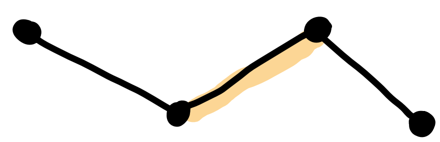
perfect

b)



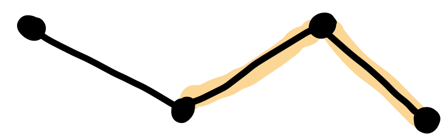
none

c)



maximal

d)



not a matching

Hall's (Marriage) Thm (3.1.11): Let G be a bipartite graph w/ parts X and Y. Then,

"partite sets"

(G is an X,Y-bigraph)

G has a matching that saturates X $\iff$ $|N(S)| \geq |S|$ for all $S \subseteq X$