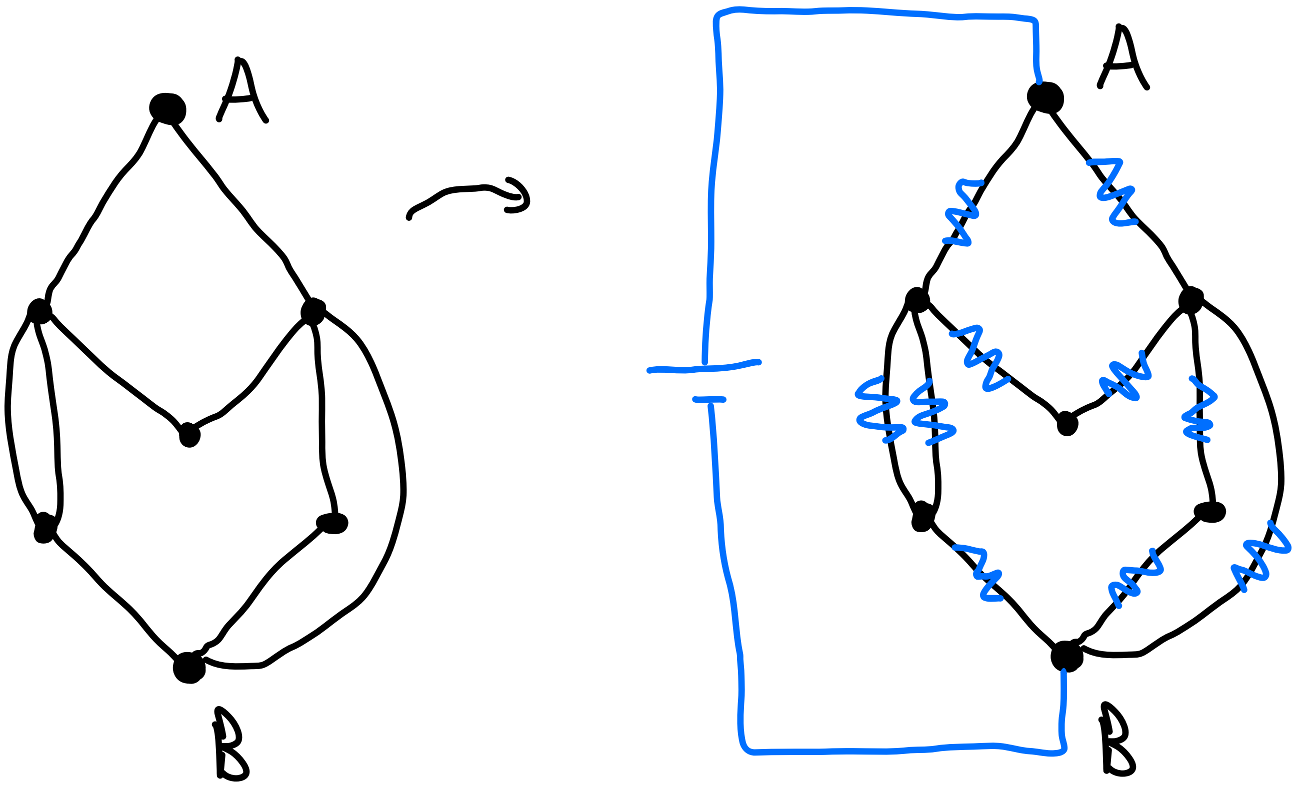
No announcements today

Recall:

Kirchoff's laws for electrical circuits

Source: Postnikov lecture notes
(link on 412 course website)

Let G be a (loopless) graph, and consider edges of G to represent resistors.
Choose vertices A and B to be connected to a source of electricity



Choose any orientation D of G

(doesn't matter which)

Quantities associated to each edge e:

- Current I_e through e
- Voltage (or potential difference) V_e across e
- Resistance R_e of e $(R_e > 0)$
- Conductance $C_e := \frac{1}{R_e}$

Three laws:

K1: At any vertex v, the sum of the in-currents equals the sum of the out-currents:

$$\sum_{\substack{e \text{ has} \\ \text{head } v}} I_e = \sum_{\substack{e \text{ has} \\ \text{tail } v}} I_e$$

K2: For any cycle C in G, the (signed) sum of voltages is 0:

$$\sum_{e \in E(C)} \pm V_e = 0,$$

where we traverse C in either direction, and the term involving V_e is positive iff we traverse e in the way it's oriented in D.

Ohm's Law: $\forall e \in E(D)$,

$$V_e = I_e R_e \qquad (I_e = V_e C_e)$$

Prop: K_2 is equivalent to the following condition:

K_2': There exists a (unique) function

$$U : V(G) \to \mathbb{R},$$

called the potential function, s.t.

a) $\forall \; u \xrightarrow{e} v$, $\quad V_e = U(v) - U(u)$

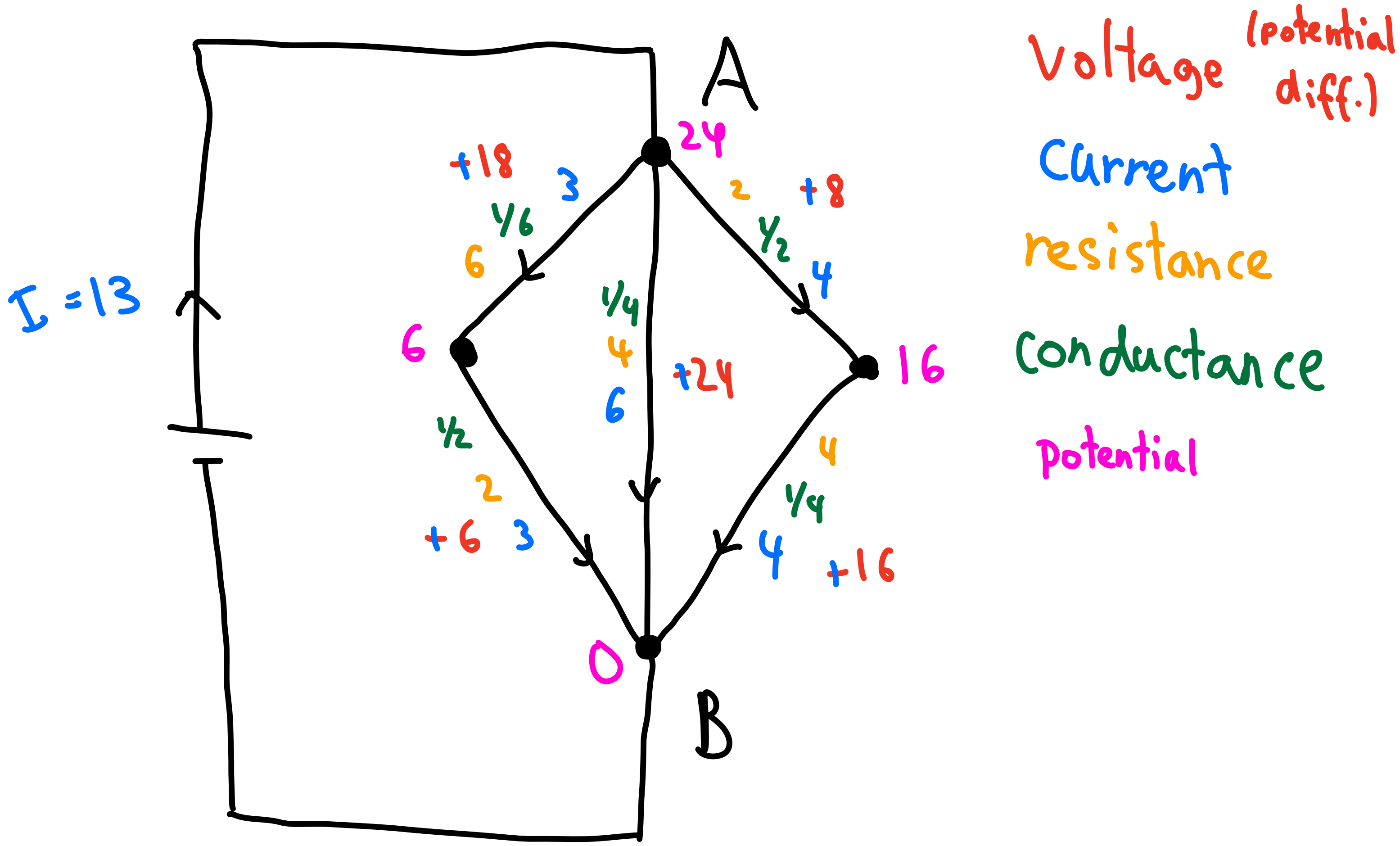
b) $U(B) = 0$

Pf: Homework!

From here on, let $V_e = U(\text{tail}) - U(\text{head})$ instead. Changes in blue

Goal: find the "effective resistance" $R(G)$ of a whole graph G

Ex:



The graph G has

total potential difference $V = 24 - 0 = 24$

resistance $R = \frac{24}{13}$

conductance $C = \frac{13}{24}$

indep. of I

Lets combine our three laws: (v fixed)

K1:
$$\sum_{\substack{e \text{ has} \\ \text{head } v}} I_e = \sum_{\substack{e \text{ has} \\ \text{tail } v}} I_e$$

Apply Ohm's Law:

$$\sum_{\substack{e \text{ has} \\ \text{head } v}} \frac{V_e}{R_e} = \sum_{\substack{e \text{ has} \\ \text{tail } v}} \frac{V_e}{R_e}$$

$$\sum_{\substack{e \text{ has} \\ \text{head } v}} V_e c_e = \sum_{\substack{e \text{ has} \\ \text{tail } v}} V_e c_e$$

Apply K_2': $V_e = U(\text{tail}) - U(\text{head})$

$$\sum_{\substack{u \xrightarrow[e]{} v \\ \text{in } D}} (U(u) - U(v))\, c_e = \sum_{\substack{v \xrightarrow[e]{} u \\ \text{in } D}} (U(v) - U(u))\, c_e$$

Rearrange:

$$\sum_{\substack{u \overset{}{\underset{e}{—}} v \\ \text{in } G}} (V(v) - V(u))\, c_e = 0$$

Actually, need to treat A, B differently:

$$\sum_{\substack{u \overset{}{\underset{e}{—}} v \\ \text{in } G}} (V(v) - V(u))\, c_e = \begin{cases} +I, & \text{if } v = A \\ -I, & \text{if } v = B \\ 0, & \text{otherwise} \end{cases}$$

Rearrange some more

$$V(v) \left(\sum_{\substack{\underset{e}{—} v \\ \text{in } G}} c_e \right) - \sum_{u} V(u) \left(\sum_{\substack{u \underset{e}{—} v \\ \text{in } G}} c_e \right) = \begin{cases} +I, & A \\ -I, & B \\ 0, & \text{else} \end{cases}$$

Surprise — this is matrix multiplication

Order $V(G)$ as $v_1 = A, v_2, \ldots, v_n = B$

Let $\vec{u} = \begin{bmatrix} U(v_1) \\ \vdots \\ U(v_n) \end{bmatrix}$ $\quad \vec{i} = \begin{bmatrix} +I \\ 0 \\ \vdots \\ 0 \\ -I \end{bmatrix}$

Then $K\vec{u} = \vec{i}$, where

$$K_{ij} = \begin{cases} \displaystyle\sum_{\substack{\overset{e}{-}v_i \\ \text{in } G}} c_e, & \text{if } i = j \\ \\ \displaystyle -\sum_{\substack{v_j \overset{e}{-} v_i \\ \text{in } G}} c_e, & \text{if } i \neq j \end{cases}$$

$K = L(G)$, the (weighted) Laplacian matrix of G!

The weight $wt(e) = c_e$, the conductance of e

How do we find the effective resistance R?

Use Ohm's Law:

$$R = \frac{V}{I} = \frac{U_1 - U_n}{I} \qquad V_i := U(v_i)$$

Shifting & scaling, take $U_n = 0$, $I = 1$, so

$$R = U_1 = \text{first entry of } L(G)^{-1} \begin{bmatrix} +1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$$

$$= \text{first entry of } L^n(G)^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

By Cramer's Rule (applied to this situation):

$$V_n = \frac{\det L^{1,n}(G)}{\det L^{n}(G)}$$

By the Matrix Tree Theorem:

$$\det L^{n}(G) = \tau(G)$$

$$\det L^{1,n}(G) = \det L^{n}(\tilde{G}) = \tau(\tilde{G})$$

where $\tilde{G} = (G \sqcup AB) \cdot AB$

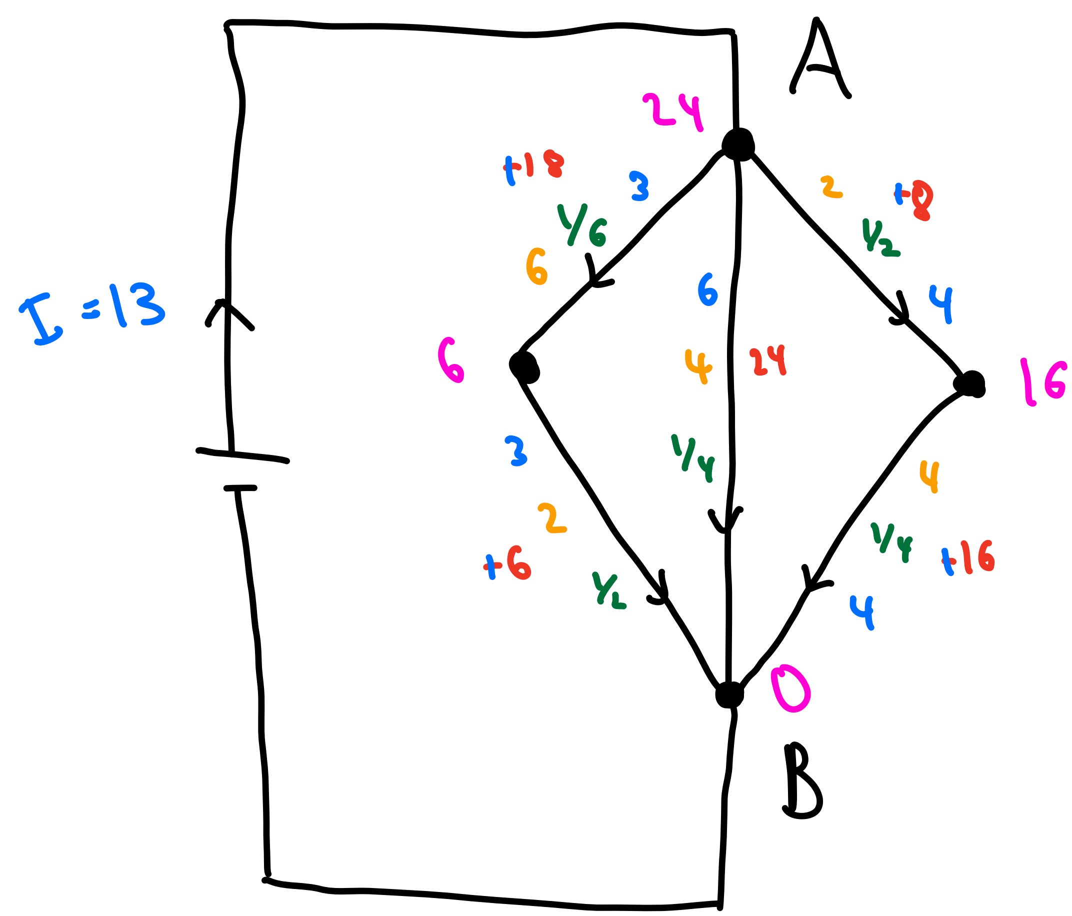
(glue A and B together)

We have proven the following:

Theorem (Kirchoff):

$$R(G) = \frac{\tau(\tilde{G})}{\tau(G)}$$

Ex:



Voltage (potential diff.)

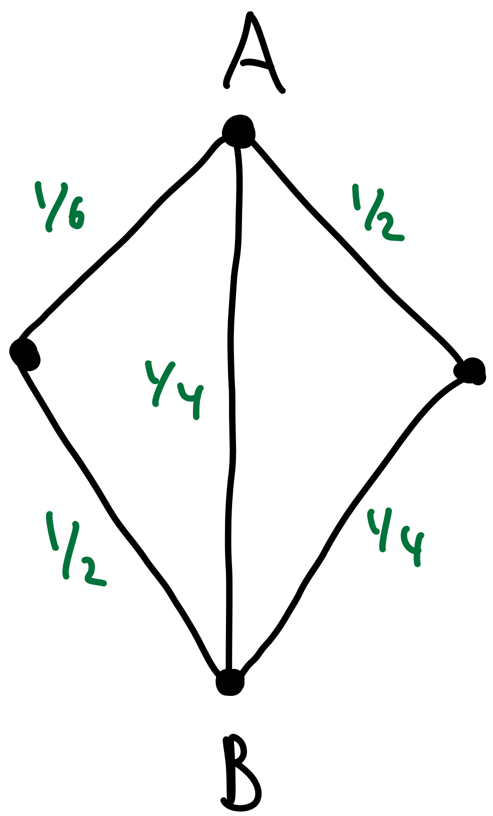
Current

resistance

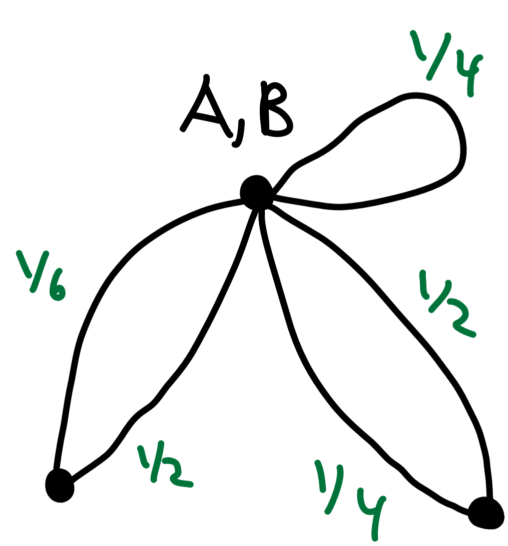
conductance

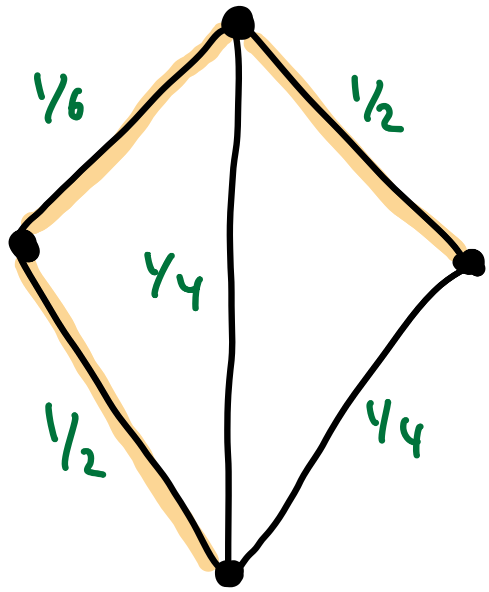
potential

G:

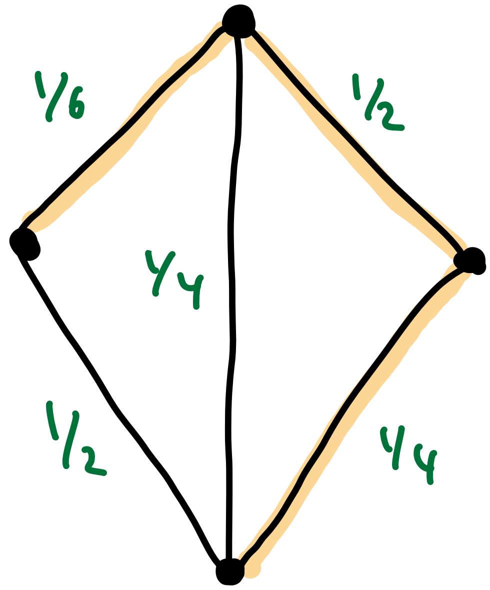


$\tilde{G}$:

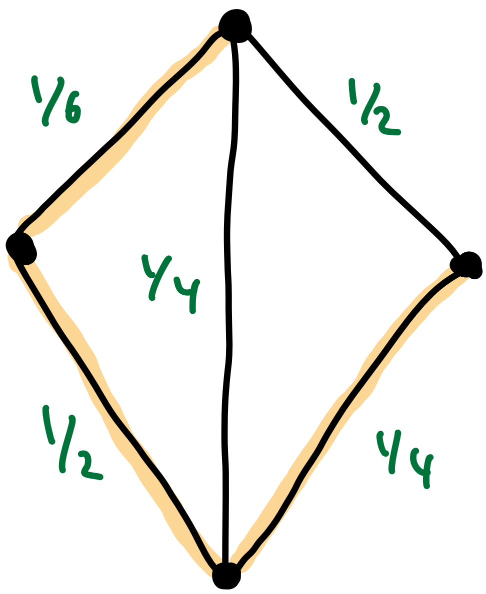




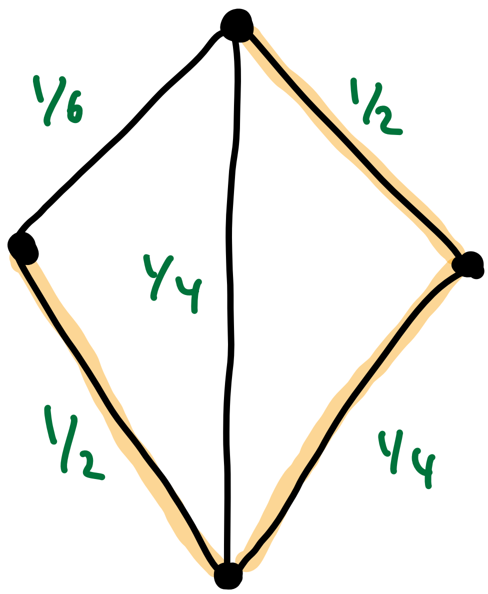
$\frac{1}{24}$



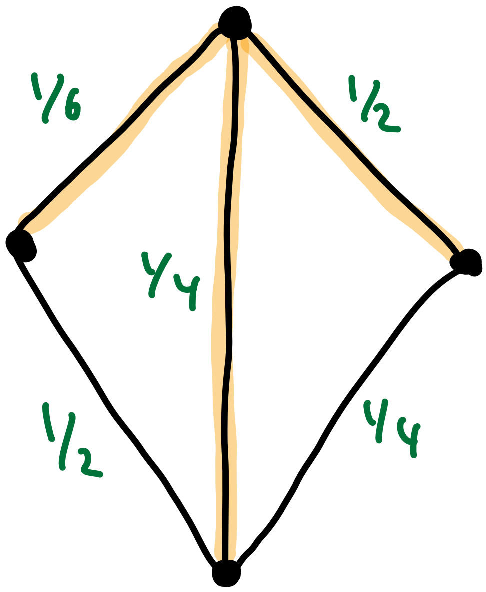
$\frac{1}{48}$



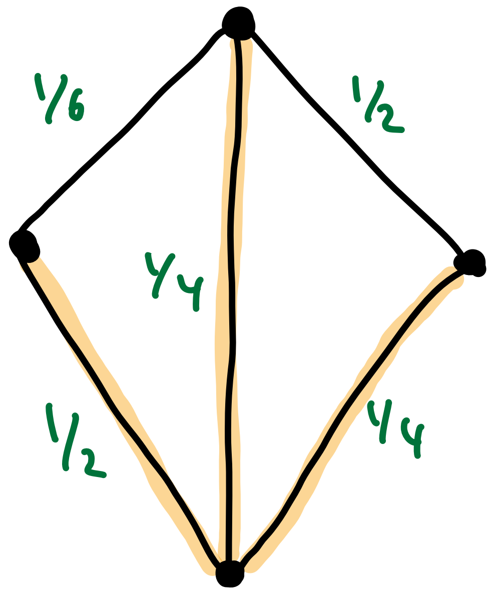
$\frac{1}{48}$



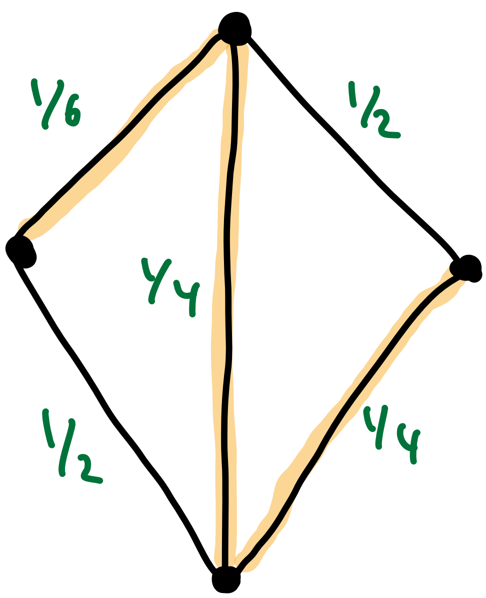
$\frac{1}{16}$



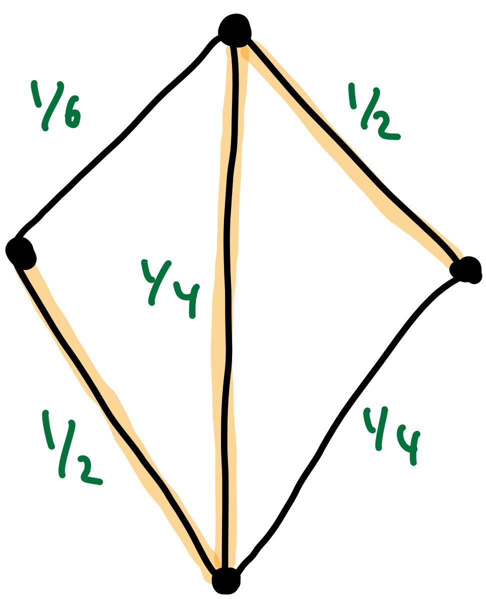
$\frac{1}{48}$



$\frac{1}{32}$

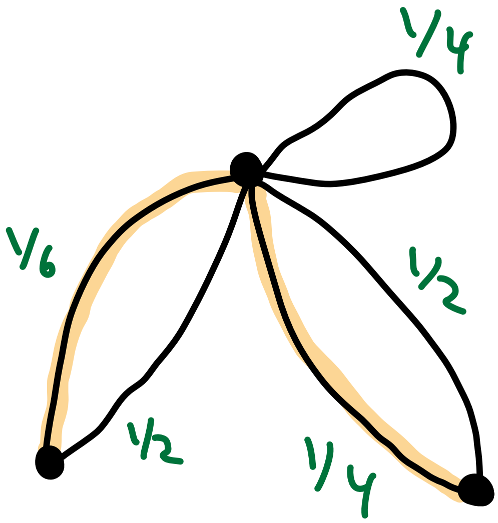


$\frac{1}{96}$

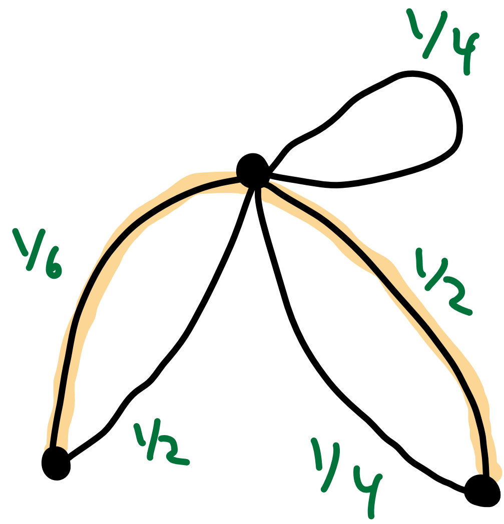


$\frac{1}{16}$

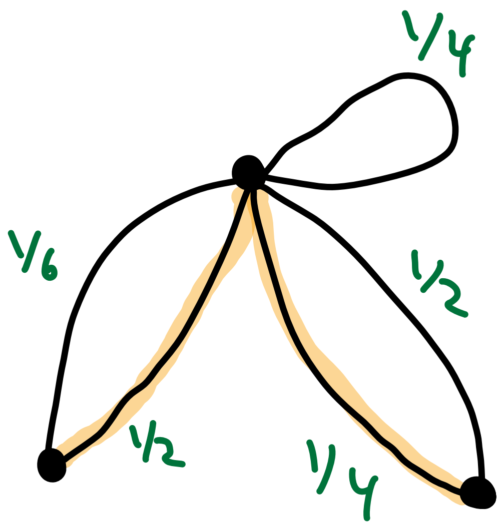
$$\tau(G) = \frac{13}{48}$$



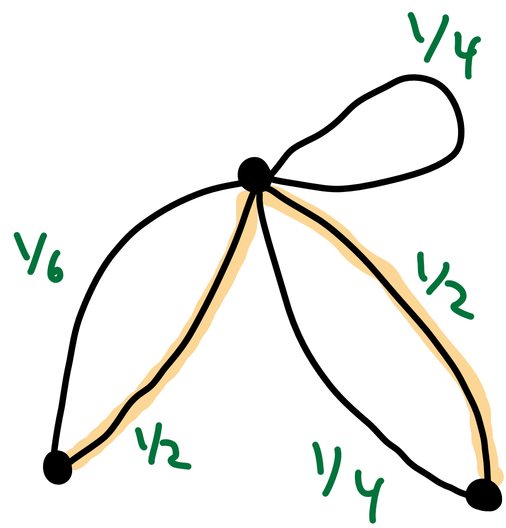
$\frac{1}{24}$



$\frac{1}{12}$



$\frac{1}{8}$



$\frac{1}{4}$

$$\tau(\tilde{G}) = \frac{1}{2}$$

$$R(G) = \frac{\tau(\tilde{G})}{\tau(G)} = \frac{24}{13} \quad \checkmark$$