Announcements: HW5 posted

Exam 1 graded

Median: 75/95

Mean: 71.4/95

Std. dev.: 17.3

Regrade request deadline: next Wed. (10/4)

Recall: Matrix tree thm.: For any loopless graph G, and for any i,

$$\tau(G) = \det \underbrace{L^i(G)}_{\text{reduced Laplacian}},$$

where $L(G) = D(G) - A(G)$ is the Laplacian matrix of G.

Pf (Godsil-Royle, Algebraic Graph Theory):

Induction on $|E(G)|$, using Prop. 2.2.8:

$$\tau(G) = \tau(G \setminus e) + \tau(G \cdot e)$$

Base case: no edges:

$$T(G) = \begin{cases} 1, & n=1 \\ 0, & n>1 \end{cases} = \det L^i(G). \checkmark$$

Inductive step:

Let e be an edge with endpoints v_i and v_j.

$$\text{Let } E = \begin{array}{c|c} & \begin{matrix} \cdots & i & \cdots & j & \cdots \end{matrix} \\ \hline \begin{matrix} \vdots \\ i \\ \vdots \\ j \\ \vdots \end{matrix} & \begin{bmatrix} & & & & \\ & 1 & & -1 & \\ & & & & \\ & -1 & & 1 & \\ & & & & \end{bmatrix} \end{array} \qquad E' = E^i$$

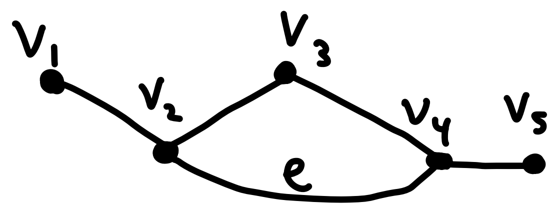
remove row/col i

Then, $L(G) = L(G \setminus e) + E$, so

$$L^i(G) = L^i(G \setminus e) + E'$$

Since E' just has a single nonzero entry, expanding along row j gives

$$\det L^i(G) = \det L^i(G \setminus e) + \det L^{i,j}(G \setminus e)$$
$$= \det L^i(G \setminus e) + \det L^{i,j}(G) \quad (*)$$



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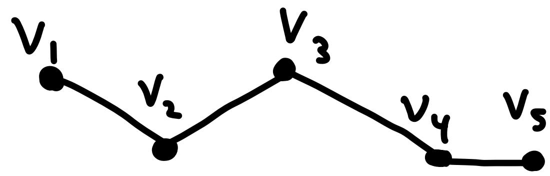


$G \setminus e$

$$\underset{L(G)}{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}} \quad \underset{L(G \setminus e)}{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}} \quad \underset{E}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

$$\underset{L^2(G)}{\begin{array}{c|c} & \begin{matrix} v_1 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}} \quad \underset{L^2(G \setminus e)}{\begin{array}{c|c} & \begin{matrix} v_1 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}} \quad \underset{E'}{\begin{bmatrix} & \\ & 1 \end{bmatrix}}$$

$$\det L^2(G) = \det L^2(G \setminus e) + 1 \cdot \det L^{2,4}(G \setminus e)$$

Return to proof:

When forming $G \cdot e$, consider the combined vertex to be labelled v_i. All edges incident to v_i and/or v_j in G are represented by row/col i in $L(G \cdot e)$.

Therefore, deleting row/col. i from $L(G \cdot e)$ is equivalent to deleting rows/cols. i <u>and</u> j from $L(G)$ i.e.

$$L^i(G \cdot e) = L^{i,j}(G),$$

so $(*)$ becomes

$$\det L^i(G) = \det L^i(G \setminus e) + \det L^i(G \cdot e)$$

By the inductive hyp.,

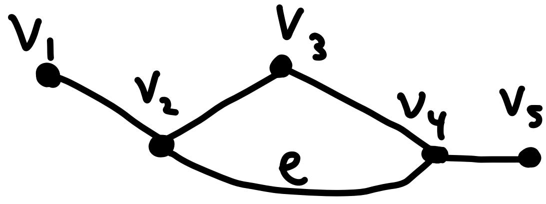
$$\det L^i(G \setminus e) = \tau(G \setminus e)$$

$$\det L^i(G \cdot e) = \tau(G \cdot e),$$

so by Prop. 2.2.8,

$$\det L^i(G) = \tau(G \setminus e) + \tau(G \cdot e) = \tau(G) \qquad \square$$

Return to example:



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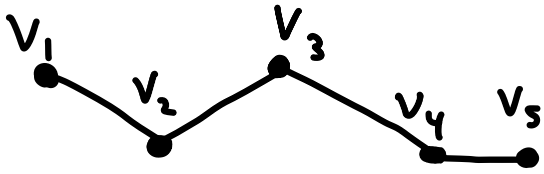
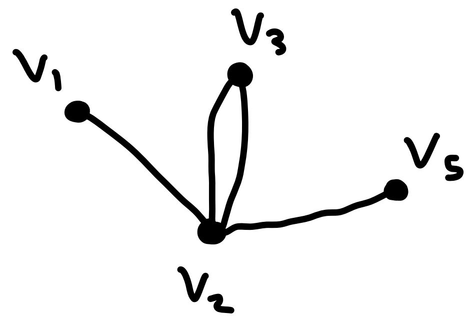


$G \setminus e$



$G \cdot e$

$$\begin{array}{c|cccc} & v_1 & v_3 & v_4 & v_5 \\ \hline v_1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 2 & -1 & 0 \\ v_4 & 0 & -1 & 3 & -1 \\ v_5 & 0 & 0 & -1 & 1 \end{array}$$

$L^2(G)$

$$\begin{array}{c|cccc} & v_1 & v_3 & v_4 & v_5 \\ \hline v_1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 2 & -1 & 0 \\ v_4 & 0 & -1 & 2 & -1 \\ v_5 & 0 & 0 & -1 & 1 \end{array}$$

$L^2(G \setminus e)$

$$\begin{array}{c|ccc} & v_1 & v_3 & v_5 \\ \hline v_1 & 1 & 0 & 0 \\ v_3 & 0 & 2 & 0 \\ v_5 & 0 & 0 & 1 \end{array}$$

$L^2(G \cdot e)$

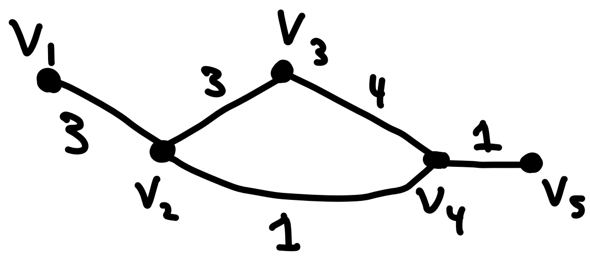
$$\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \quad 1 \quad + \quad 2 \quad = 3$$

There are many generalizations of the Matrix Tree Theorem. Here's one:

Def:

a) A weighted graph G is a graph together with a function $wt: E(G) \to \mathbb{R}$

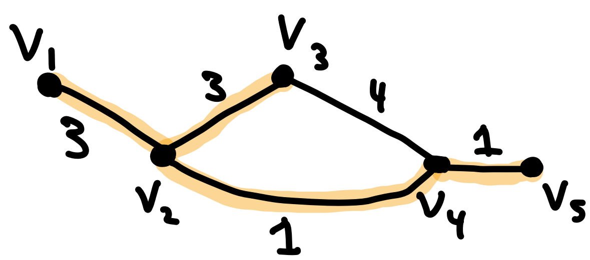


$wt(v_1 v_2) = 3$
$\vdots$

b) If T is a spanning tree of G, the weight of T is

$$wt(T) := \prod_{e \in E(T)} wt(e)$$

T



$wt(T) = 3 \cdot 3 \cdot 1 \cdot 1 = 9$

c) The tree sum $\tau(G)$ of G is

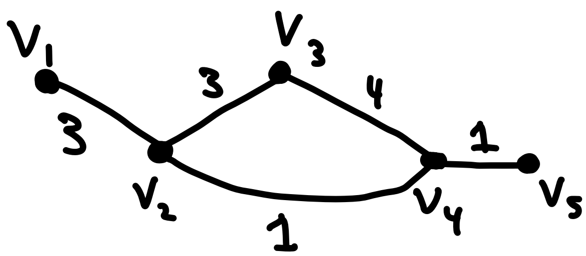
$$\tau(G) = \sum_{T \text{ sp. tree of } G} wt(T)$$

d) The (weighted) Laplacian matrix $L(G)$ of G is given by:

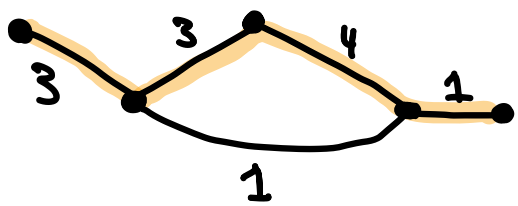
$$L_{ij} = \begin{cases} \displaystyle\sum_{\substack{e \text{ incident} \\ \text{to } i}} wt(e), & \text{if } i = j \\ \displaystyle -\sum_{\substack{e \text{ has} \\ \text{endpoints } i,j}} wt(e), & \text{if } i \neq j \end{cases}$$

Class activity:

Find $\tau(G)$ and $L(G)$ for:



$\tau(G) = 57$



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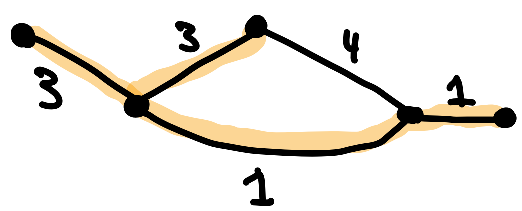

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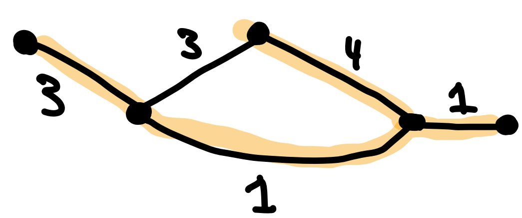

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$$\begin{bmatrix} 3 & -3 & 0 & 0 & 0 \\ -3 & 7 & -3 & -1 & 0 \\ 0 & -3 & 7 & -4 & 0 \\ 0 & -1 & -4 & 6 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$L(G)$

Weighted Matrix Tree Theorem: For any loopless weighted graph G and any i,

$$\tau(G) = \det L^i(G)$$

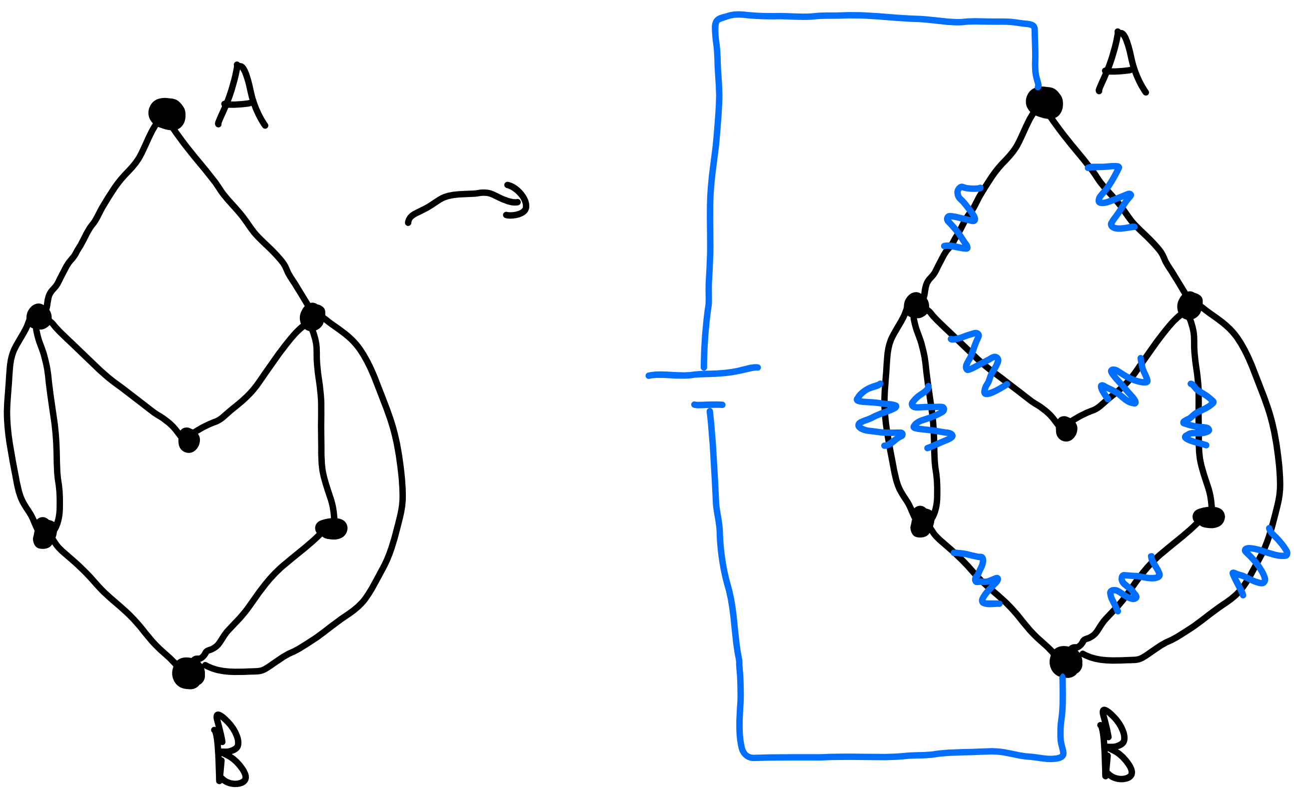
Pf: Homework!

Application/motivation:

Kirchoff's laws for electrical circuits

Source: Postnikov lecture notes (link on 412 course website)

Let G be a (loopless) graph, and consider edges of G to represent resistors.
Choose vertices A and B to be connected to a source of electricity



Choose any orientation D of G

(doesn't matter which)

Quantities associated to each edge e:

- Current I_e through e
- Voltage (or potential difference) V_e across e
- Resistance R_e of e $(R_e > 0)$
- Conductance $C_e := \frac{1}{R_e}$

Three laws:

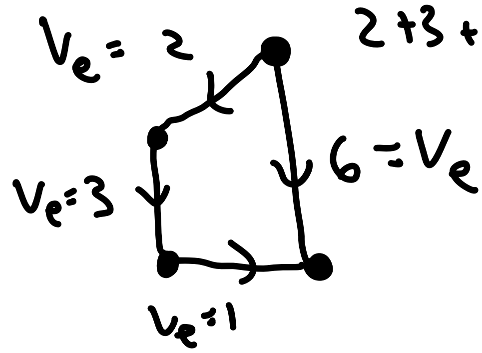
K1: At any vertex v, the sum of the in-currents equals the sum of the out-currents:

$$\sum_{\substack{e \text{ has} \\ \text{head } v}} I_e = \sum_{\substack{e \text{ has} \\ \text{tail } v}} I_e$$

K2: For any cycle C in G, the (signed) sum of voltages is 0:

$2+3+1-6=0$

$$\sum_{e \in E(C)} \pm V_e = 0,$$



where we traverse C in either direction, and the term involving V_e is positive iff we traverse e in the way it's oriented in D.

Ohm's Law: $\forall e \in E(D)$,

$$V_e = I_e R_e \qquad (I_e = V_e C_e)$$

Prop: K_2 is equivalent to the following condition:

K_2': There exists a (unique) function

$$U : V(G) \to \mathbb{R},$$

called the potential function, s.t.

a) $\forall \; u \xrightarrow{e} v$, $\quad V_e = U(v) - U(u)$

b) $U(B) = 0$

Pf: Homework!