No announcements today

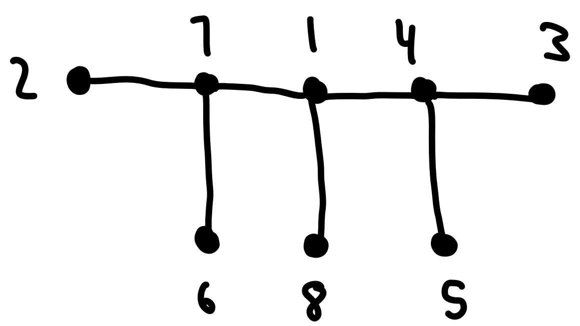
Recall:

Def: The <u>Prüfer Code</u> $f(T) = (a_1, \dots, a_{n-2})$ of T is given by the following algorithm:

At step i:

- delete the leaf w/ the smallest label
- a_i is the label for the (unique) neighbor of the leaf

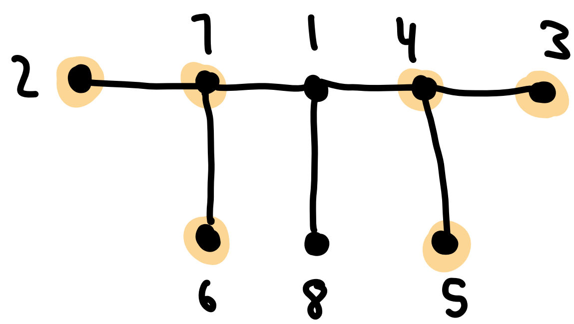
Ex:



$Pru(T) = 744171$

Can go backwards:

$Pru(T) = 744171$



Cayley's Formula (Thm 2.2.3): There are n^{n-2} labelled trees with n vertices

Pf: $n=1$ good

We prove that for $n \geq 2$

$$T \longleftrightarrow \mathrm{Pru}(T)$$

is a bijection. Then it will follow that there are n^{n-2} labelled trees since there are that many Prufer codes.

Base case: $n=2$

 $\longleftrightarrow \ \emptyset$ ← length zero

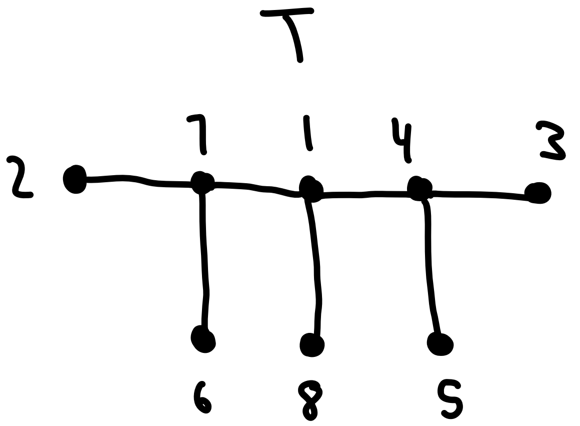
Inductive step: $n>2$.

label set of T

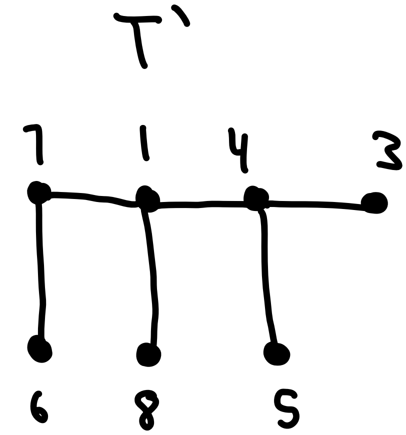
$a=(a_1,\dots,a_{n-2})$ $\qquad$ $a' = (a_2,\dots,a_{n-2})$

Let x be the min'l elt. of S not in a.

Inductive hypothesis: $\exists!\, T'$ w/ label set $S' := S \setminus x$ s.t. $\mathrm{Pru}(T') = a'$. Form T from T' by adding the vertex x and the edge xa_1. Then, $\mathrm{Pru}(T) = a$ since after step 1 of the algorithm, we have $\mathrm{Pru}(T) = (a_1, \dots)$ and the remaining tree is T'.



$Pru(T) = 744171$



$Pru(T') = 44171$

Conversely, if τ is any tree w/ $Pru(\tau) = a$, then x is its smallest leaf, so  , and when we remove this edge, we are left w/ a tree τ' which must have $Pru(\tau') = a'$. By the inductive hyp., $\tau' = T'$, so $\tau = T$. $\square$

Cor 2.2.4: Let $d_1, \dots, d_n \in \mathbb{Z}_{\geq 1}$ s.t. $d_1 + \dots + d_n = 2n-2$. Then the number of trees w/ label set $\{1, \dots, n\}$ s.t. vertex i has degree d_i is $\dfrac{(n-2)!}{\prod (d_i - 1)!}$

Pf sketch: Look at how many times i appears in $Pru(T)$

Further question: How many spanning trees does a graph G have?

$\tau(G) :=$ number of spanning trees of G

Cases we know so far:

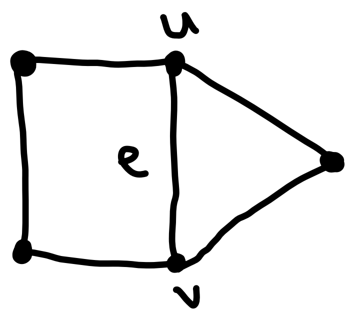
- $\tau(\text{tree}) = 1$ Def'n
- $\tau(\text{disconn. graph}) = 0$ Cor 2.1.5
- $\tau(C_n) = n$
- $\tau(K_n) = n^{n-2}$ Cayley's Formula
- $\tau(G) = \tau(G \setminus \text{loops})$

Matrix Tree Theorem (2.2.12) $\tau(G)$ can be given as the determinant of a certain matrix.

Need a recursive tool first:

Def 2.2.7: Let $e \in E(G)$ have endpoints u and v. The contraction $G \cdot e$ is the graph obtained from G by replacing u and v with a single vertex whose incident edges are the edges other than e that were incident to u or v.

Class activity: Find $G \cdot e$



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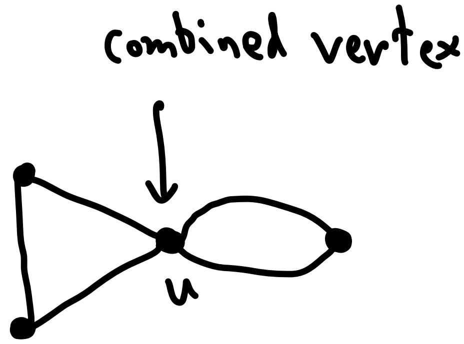


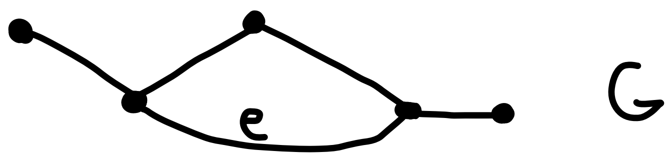
$G \cdot e$

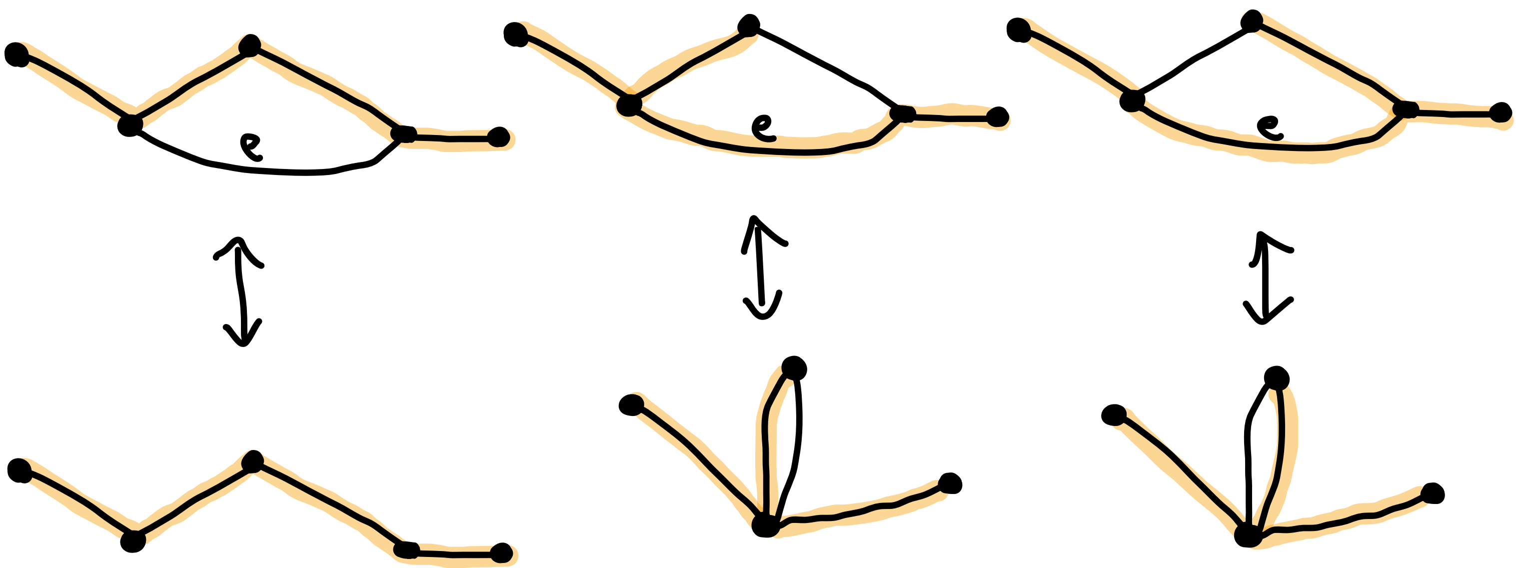
Prop 2.2.8: If e is not a loop, then

$$\tau(G) = \tau(G \setminus e) + \tau(G \cdot e)$$

"deletion - contraction"

Pf: The spanning trees of G that omit e are precisely the spanning trees of $G \setminus e$. On the other hand, if T is a spanning tree of G containing e, then $T' := T \cdot e$ is a spanning tree of $G \cdot e$, and if T' is a spanning tree of $G \cdot e$, then $T := T' \cup e$ is the unique spanning tree of G s.t. $T \cdot e = T'$. $\square$





Spanning tree of $G \setminus e$

Spanning tree of $G \cdot e$

Spanning tree of $G \cdot e$

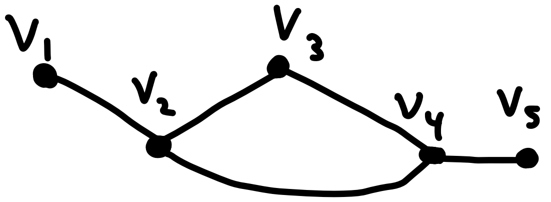
Def:

a) The <u>degree matrix</u> $D(G)$ is the diagonal matrix with (i,i)-entry equal to $d(v_i)$

b) The <u>Laplacian matrix</u> of G is the matrix

$$L(G) = D(G) - \underbrace{A(G)}_{\text{adjacency matrix}}$$

c) The <u>reduced Laplacian</u> $L^i(G)$ is $L(G)$ with the ith row and column deleted



G

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$D(G)$ $\qquad$ $A(G)$ $\qquad$ $L(G)$

$$\begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L^1(G) \qquad\qquad L^2(G) \qquad\qquad L^{1,2}(G)$$

Matrix Tree Theorem: For any loopless graph G, and for any i,

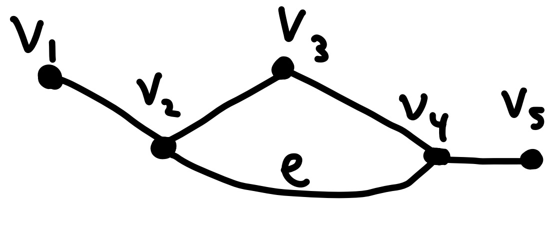
$$\tau(G) = \det L^i(G)$$

Pf (Godsil-Royle, Algebraic Graph Theory):

Induction on $|E(G)|$, using Prop. 2.2.8.

Base case: no edges:

$$\tau(G) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases} = \det L^i(G)$$



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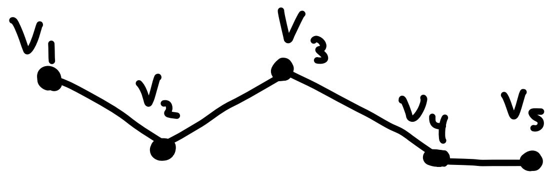

$G \setminus e$

$$L(G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad L(G \setminus e) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$L^2(G) = \begin{array}{c} \\ v_1 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \qquad L^2(G \setminus e) = \begin{array}{c} \\ v_1 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \qquad E' = \begin{bmatrix} & & & \\ & & & \\ & & & 1 \\ & & & \end{bmatrix}$$

$$\det L^2(G) = \det L^2(G \setminus e) + 1 \cdot L^{2,4}(G \setminus e)$$