Announcements

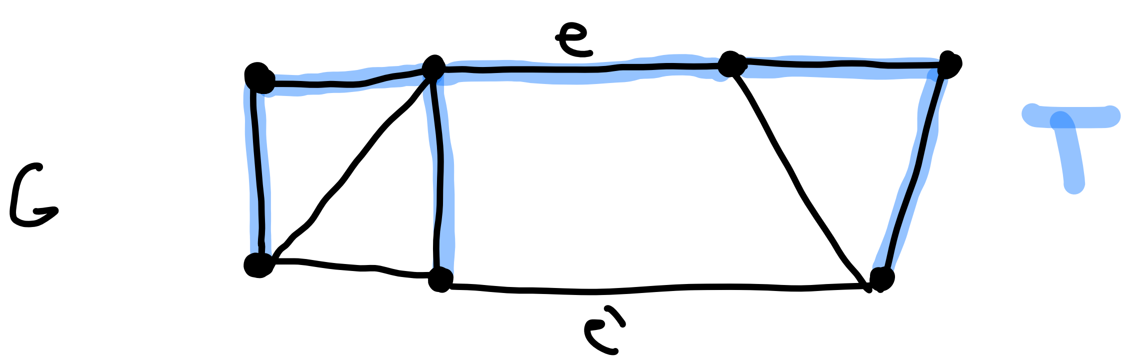
HW4 posted (due Wed. 9/27)

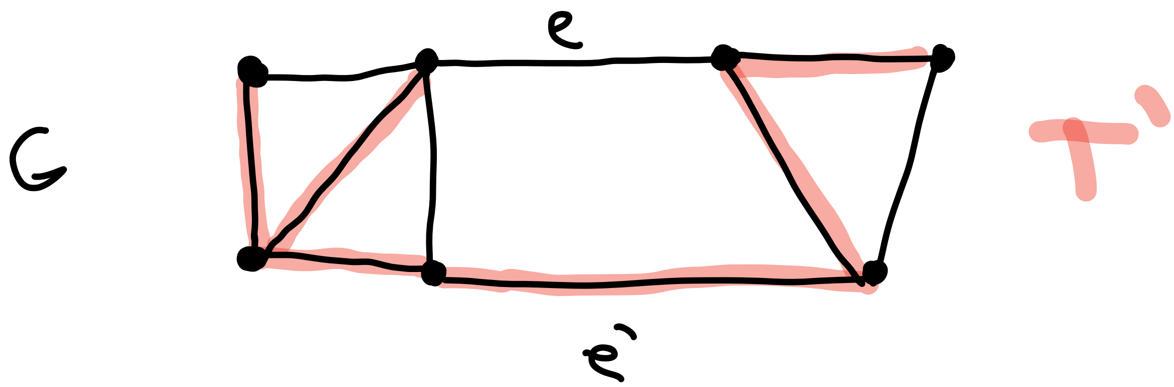
Prop (2.1.6/2.1.7): Let G be a graph w/ spanning trees T, T'.

a) For all $e \in E(T)$, $\exists e' \in E(T')$ s.t. $(T \cup e') \setminus e$ is a spanning tree of G.

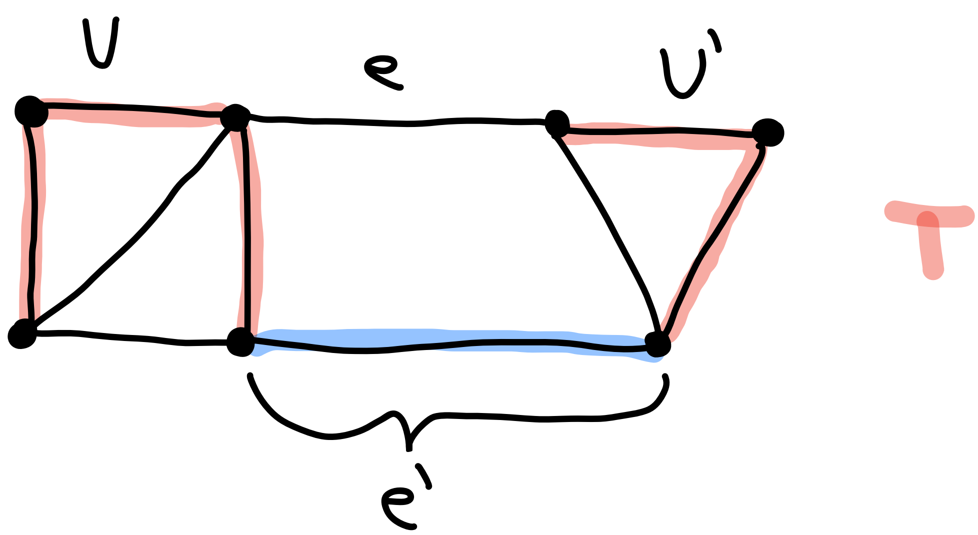
b) For all $e' \in E(T')$, $\exists e \in E(T)$ s.t. $(T \cup e') \setminus e$ is a spanning tree of G.

[If you tell me which edge to remove, I'll tell you which edge to add
If you tell me which edge to add, I'll tell you which edge to remove]





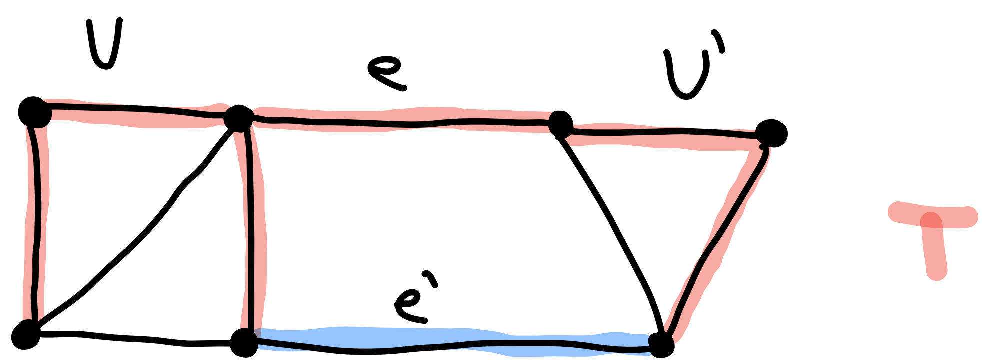
Pf: a) By def'n (e) (2.1.4) of a tree, every edge of T is a cut-edge. So let U and U' be the two components of $T \setminus e$. Since T' is connected, $\exists e' \in E(T')$ with endpoints in U and U', so $(T \setminus e) \cup e'$ is connected, has vertex set $V(G)$ and has $|E(T)| = n(G) - 1$ edges, so by def'n (c), $(T \setminus e) \cup e'$ is a spanning tree of G.



b) If $T = T'$, let $e = e'$. Otherwise,

$T \cup e'$ has $n(G)$ edges, it is not a tree, by def'n (c), so by def'n (a), it has a cycle C. C is the unique cycle since $(T \cup e') \setminus e' = T$ is a tree and hence acyclic. In addition, C contains ≥ 2 edges so let $e \in E(C)$ s.t. $e \neq e'$. By Thm. 1.2.14, e is not a cut edge since it belongs to a cycle, so $(T \setminus e) \cup e'$ is a conn. graph w/ $n(G)-1$ edges and vertex set $V(G)$; hence a spanning tree.

[Note the choice for e]



Def 2.1.9/12: G: graph

a) The distance $d(u,v)$ from u to v is the shortest length of a u,v-path (∞ if no path)

b) The diameter $\operatorname{diam} G$ is the maximum distance btwn. any two vertices in G (∞ if disconn.)

$$\operatorname{diam} G = \max_{u,v \in V(G)} d(u,v)$$

c) The eccentricity of a vertex u is

$$\varepsilon(u) := \max_{v \in V(G)} d(u,v)$$

(i.e. $\operatorname{diam} G = \max_{u \in V(G)} \varepsilon(u)$)

d) The radius of G is

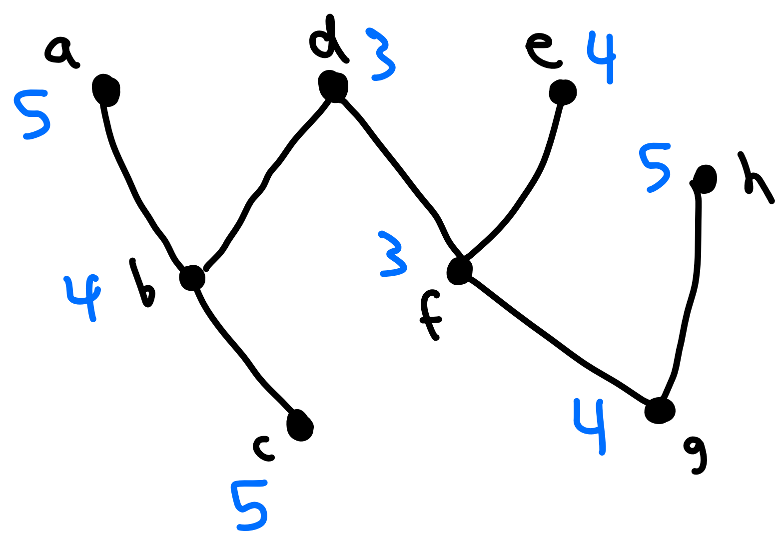
$$\operatorname{rad} G := \min_{u \in V(G)} \varepsilon(u)$$

e) The center of G is the induced subgraph

$$G\left[\{u \in V(G) \mid \varepsilon(u) = \operatorname{rad}(u)\}\right]$$

Class activity:

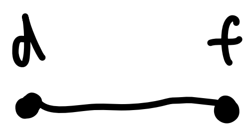
Find diam G, rad G, the eccentricity of each vertex, and the center



rad G = 3

diam G = 5

center G =



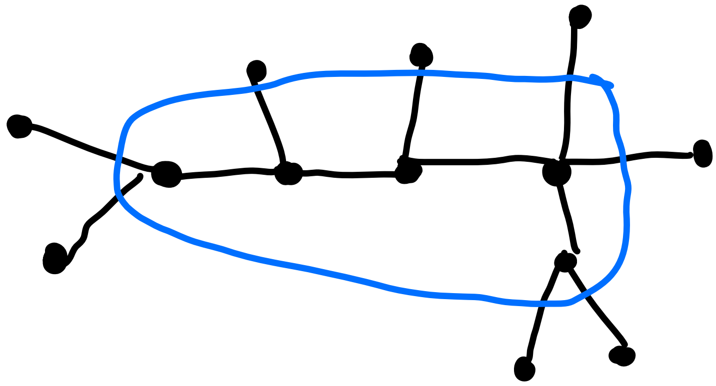
Jordan Tree Theorem: The center of a tree is • or •—•

Pf sketch: induction on $n := n(T)$

$n \le 2$: the center is the entire tree • or •—•

$n > 2$: Let $T' := T[\text{non-leaves}]$. By def'n C

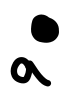
(2.1.4), T' is a tree.



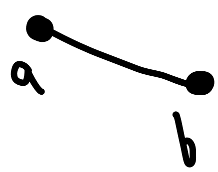
If $u \in V(T')$, $\varepsilon_T(u) = \varepsilon_{T'}(u) + 1$, if u is a leaf of T, it's not in the center of T. Thus, the center of T equals the center of T'. $\square$

How many (labelled) n-vertex trees are there?

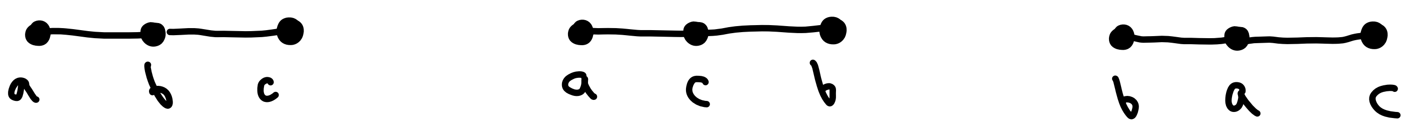
$n=1$:



$n=2$:



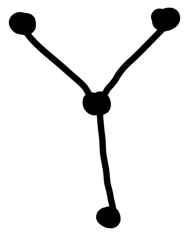
$n=3$:



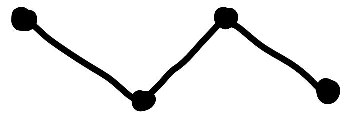
$$\begin{bmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{bmatrix} \qquad \begin{bmatrix} & & 1 \\ & & 1 \\ 1 & 1 & \end{bmatrix} \qquad \begin{bmatrix} & 1 & 1 \\ 1 & & \\ 1 & & \end{bmatrix}$$

$n = 4$:

Total: 16

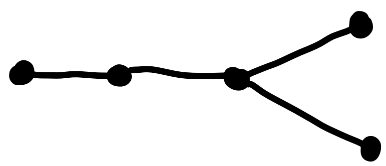


$\times 4$

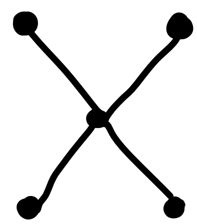


$\times 12$

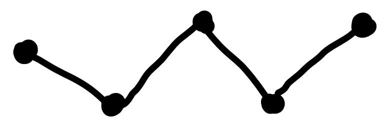
$n = 5$:



$\times 60$



$\times 5$



$\times 60$

Total: 125

Pattern?

Cayley's Formula (Thm 2.2.3): There are n^{n-2} labelled trees with n vertices

[For technical reasons, label set is some $S \subseteq \mathbb{N}$]

Pf idea:

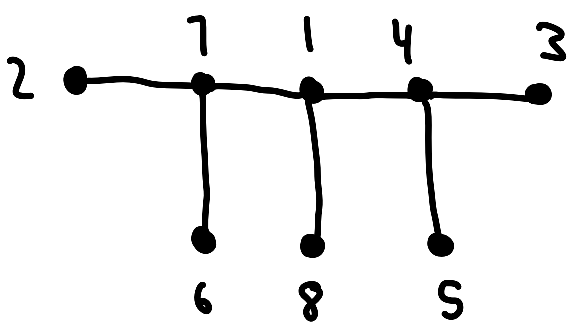
n-vertex trees $\longleftrightarrow$ length $n-2$ strings of elts. of S

Def: The <u>Prüfer Code</u> $\text{Pru}(T) = (a_1, \ldots, a_{n-2})$ of T is given by the following algorithm:

At step i:

- delete the leaf w/ the smallest label
- a_i is the label for the (unique) neighbor of the leaf

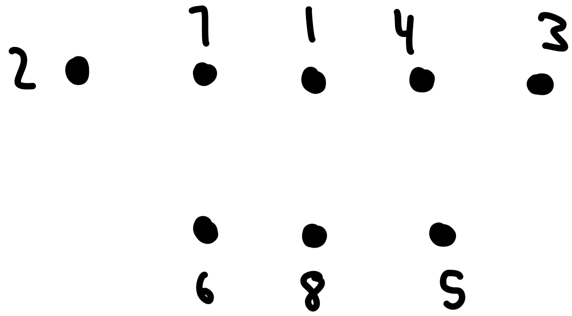
Ex:



$\text{Pru}(T) = 744171$

Can go backwards:

$\mathrm{Pru}(T) = 744171$

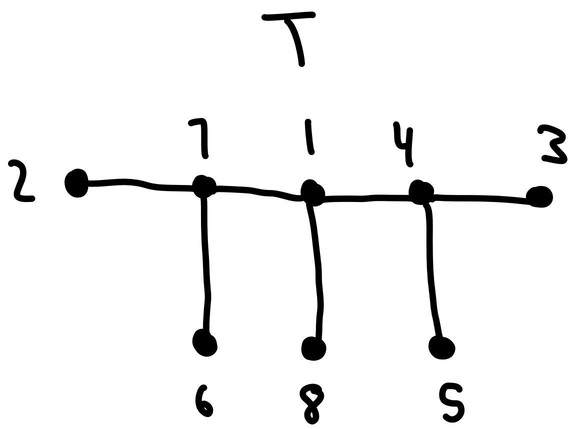


Pf of Cayley's Formula: $n=1$ good

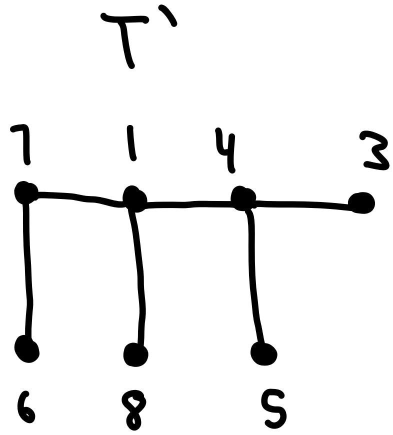
We prove that for $n \geq 2$

$$T \longleftrightarrow \mathrm{Pru}(T)$$

is a bijection.



$Pru(T) = 744171$



$Pru(T') = 44171$

Cor 2.2.4: Let $d_1, \dots, d_n \in \mathbb{Z}_{\geq 1}$ s.t. $d_1 + \dots + d_n = 2n-2$. Then the number of trees w/ label set $\{1, \dots, n\}$ s.t. vertex i has degree d_i is $\dfrac{(n-2)!}{\prod (d_i - 1)!}$