Announcements:

- Midterm 1 tonight! 7:00-8:30pm (Noyes 217)
  - Topics: All of chapter 1
  - Reference sheet allowed (two-sided)
  - See last week's email for full policies

Today: Review

Def'ns: (too many to list)

Big theorems:

Eulerian circuits/trails for graphs/digraphs

Mantel's Theorem (max. edges in Δ-free graph)

Konig's Theorem (bipartite $\Leftrightarrow$ no odd cycles)

Havel-Hakimi Theorem

Important graph examples: complete graph K_n, cycle C_n

complete bipartite graph $K_{r,s}$, hypercube Q_k,

Petersen graph, de Bruin digraph

Proof techniques to keep in mind:

Extremality

Induction

Counting

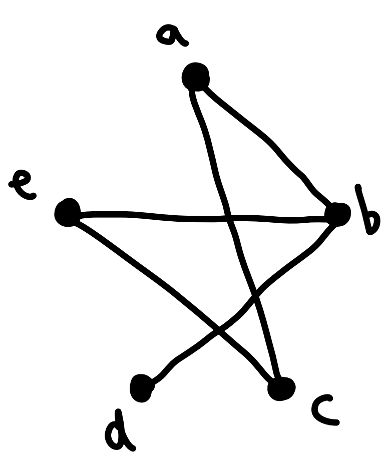
Examples:

1) Isomorphism: Determine which of the following graphs are isomorphic.

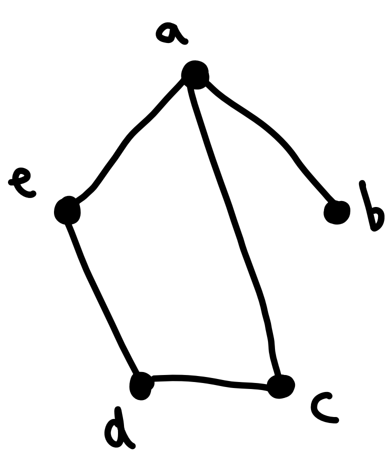
Methods to prove graphs <u>aren't</u> isomorphic:

- Degree sequence (e.g. # edges, largest degree)
- Subgraphs (e.g. cycles, induced subgraphs)
- Bipartiteness / connectivity / longest path / etc.
- Trace / determinant of adjacency matrix (not advised)

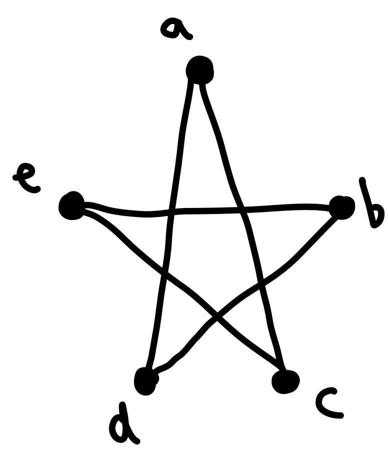
G)



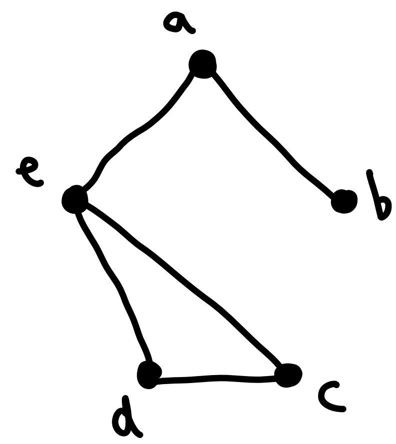
H)



K)



L)



G, H, & L all have a vertex of degree 1, and K doesn't, so K is not isom. to any of the others.

G, H, L have same deg. seq. (3, 2, 2, 2, 1)

L has a 3-cycle, while G and H don't, so L isn't isom. to the others.

Let $f: V(G) \to V(H)$ be the following bijection:

$f(a) = c$ $\quad$ $f(c) = d$ $\quad$ $f(e) = e$

$f(b) = a$ $\quad$ $f(d) = b$

Then we show that f is an isomorphism i.e. that $uv \in E(G) \Leftrightarrow f(u)f(v) \in E(H)$

$$ab \in E(G) \leftrightarrow f(a)f(b) = ca \in E(H)$$

$$ac \in E(G) \leftrightarrow f(a)f(c) = cd \in E(H)$$

$$bd \in E(G) \Leftrightarrow f(b)f(d) = ab \in E(H)$$

$$be \in E(G) \longleftrightarrow f(b)f(e) = ae \in E(H)$$

$$ce \in E(G) \Leftrightarrow f(c)f(e) = de \in E(H)$$

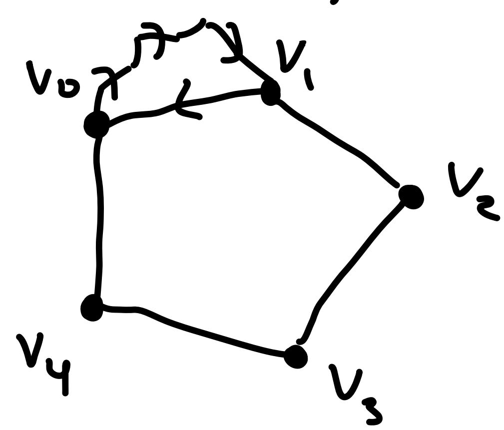
Since both graphs have exactly 5 edges, we are done. $\square$

2) Digraphs.

Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle.

Pf: Let G have the cycle $C: v_0, v_1, \dots, \underset{\overset{\shortparallel}{v_0}}{v_k}$, where k is odd, and let D be an orientation of G that is strongly connected.

Since D is strongly connected, for all i, $\exists$ a v_i, v_{i+1}-path in D.



If, for a given i, all such paths are even length, then we must have $v_i \xleftarrow{e} v_{i+1}$ (otherwise this is an odd path), and taking e followed by any v_i, v_{i+1}-path forms an odd cycle.

Therefore, assume that for all i, there exists an odd v_i, v_{i+1}-path P_i. Then the path $P_0 P_1 P_2 \cdots P_{k-1}$ (concatenate these paths) is a closed odd walk, which by Lemma 1.2.15 contains an odd cycle. $\square$

a lemma proved in class: "every closed odd walk contain an odd cycle"

3) Havel - Hakimi Theorem

Determine whether the following sequence is graphic, and if so, draw a graph with that as its deg. seq.

$$d = (5, 5, 2, 2, 2, 1, 1, 1, 1, 0)$$

We apply the Havel-Hakimi Theorem. Let $d_0 = d$, and for all $i \geq 1$, let $d_i = d_{i-1}'$, where d' refers to the corresponding sequence from H-H.

Then we have:

$$d_0 = (5, 5, 2, 2, 2, 1, 1, 1, 1, 0)$$

$$d_1 = (4, 1, 1, 1, 0, 1, 1, 1, 0) = (4, 1, 1, 1, 1, 1, 1, 0, 0)$$

$$d_2 = (0, 0, 0, 0, 1, 1, 0, 0) = (1, 1, 0, 0, 0, 0, 0, 0)$$

$$d_3 = (0, 0, 0, 0, 0, 0, 0)$$ can stop here

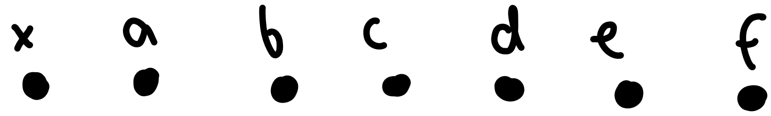
$$d_4 = (0, 0, 0, 0, 0, 0)$$

$$\vdots$$

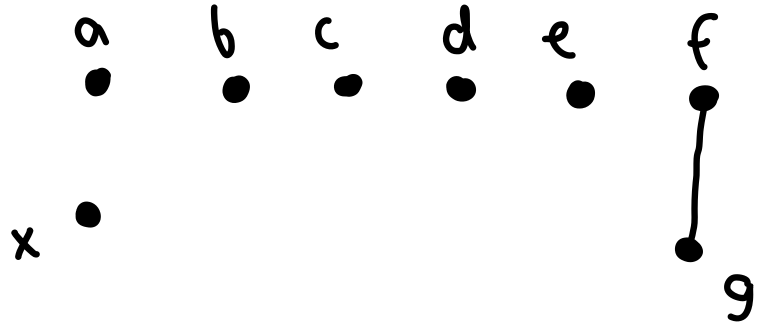
$$d_9 = (0)$$

graphic!

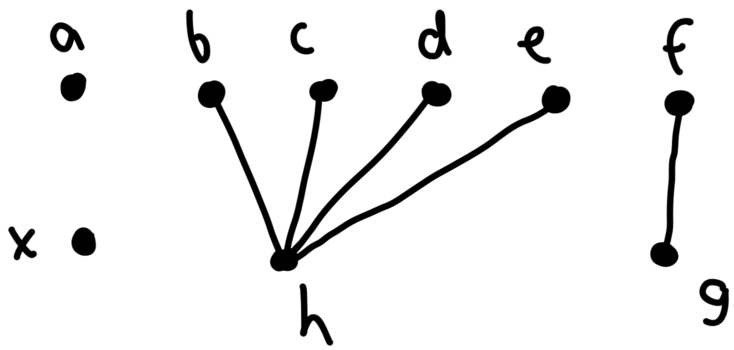
$d_3 = (0,0,0,0,0,0,0)$



$d_2 = (1,1,0,0,0,0,0,0)$



$d_1 = (4,1,1,1,1,1,1,0,0)$



$d = d_0 = (5,5,2,2,2,1,1,1,1,0)$