Math 412 Midterm Exam 1 — Sept. 20, 2023

Full Name: ____

- Complete the following problems. In order to receive full credit, please give complete arguments and/or proofs.
- You may cite without proof any result proved in class and any result from Chapter 1 of West. You may also use the results in the statements of the homework problems, including problems that you didn't complete. You may NOT use (without proving them first) any other results, including results from later in the textbook or exercises that were not assigned as homework problems.
- When citing a result, it is not necessary to give a theorem number (although you may), but you should make clear what result you're using.
- All of these instructions may be superseded by the instructions on a specific problem.
- Please check that your copy of this exam contains 5 pages of exam questions, *numbered* in the upper-right, and that it is adequately stapled.
- You may use 1 handwritten piece of $8.5'' \times 11''$ paper (both sides) with formulas and other notes as a "reference sheet". No electronic devices, including phones, headphones, smartwatches, or calculation aids, are permitted for any reason.
- **You have 1.5 hours.** The instructor will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to the instructor.
- Paper not provided by the instructor (apart from your own reference sheet) is prohibited. If you need extra room for your answers, use one of the blank pages provided (those pages except for the one at the end are labeled at the bottom by lower-case Roman numerals, starting with "ii"), and **clearly indicate that that your answer continues there**. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination and have abided by all exam policies mentioned on this page."

Signature: ________________________________

Announcements:

- Midterm 1: Wed. 9/20 7:00-8:30pm (Noyes 217)
  - Topics: All of chapter 1
  - Reference sheet allowed (two-sided)
  - See Monday's email for full policies
- Tuesday problem session (4:00-5:30pm) will be a review session (bring your own Q's)
- Wednesday class will be review (I'll bring the Q's)

Chapter 2: Trees & Distance

Thm 2.1.4: Let G be a graph w/ $|V(G)| = n$
The following are equivalent:

a) G is connected and has no cycles

b) G is connected and has $n-1$ edges

c) G has $n-1$ edges and no cycles

d) $\forall u,v \in V(G)$ $\exists!$ u,v-path in G and G is loopless (unique)

e) G is connected and every edge is a cut edge

Pf: a) $\Rightarrow$ d): Since G is conn., $\forall\, u,v \in V(G)$, $\exists\, u,v$-path. If there are vertices $u,v \in V(G)$ w/ (at least) two distinct u,v-paths P_1 and P_2, let $W = P_1 \overleftarrow{P_2}$ be the closed walk formed by traversing P_1 followed by the reverse of P_2.

Since $P_1 \neq P_2$, let e be an edge in one of P_1, P_2, but not both. Then e appears exactly once (i.e. an odd # of times) in W, so by HW2 #2, W contains a cycle (containing e).

d) $\Rightarrow$ e) Since $\forall\, u,v \in V(G)$ $\exists\, u,v$-path in G, G is conn. Let $e \in E(G)$ and let u,v be it's endpoints. Then, u,e,v is a path from u to v, and by assumption, it is unique. Thus, $G \setminus e$ has no path from u to v i.e. $G \setminus e$ is disconnected, so e is a cut edge.

e) $\Rightarrow$ a) This follows from Thm. 1.2.14: "An edge is a cut-edge iff it belongs to no cycle"

a) $\Rightarrow$ b) If $n=1$, this is clear. Otherwise, suppose G is a minimal-order connected acyclic graph w/ $|E(G)| \neq |V(G)| - 1$. By Lemma 1.2.25, since G is conn. and acyclic (no cycles), $\exists v \in V(G)$ w/ $d(v) = 1$. Then,

$G' := G \setminus v$ has one fewer vertex and one fewer edge than G, so G' is a smaller-order conn. acyclic graph w/ $|E(G')| \neq |V(G')| - 1$, contradicting the minimality of G.

b) $\Rightarrow$ c) Again using Thm. 1.2.14, if e is contained in a cycle of G, then e is not a cut-edge, so $G \setminus e$ is conn. w/ $|E(G \setminus e)| = |V(G \setminus e)| - 2$. However, this contradicts Prop. 1.3.13.

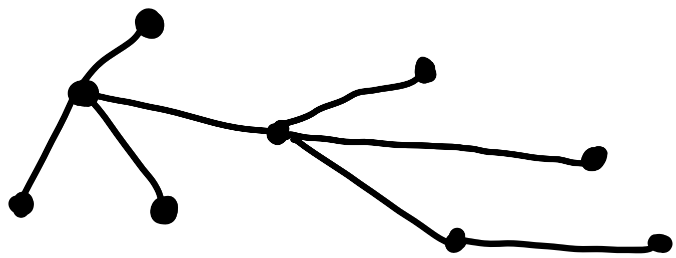
c) $\Rightarrow$ a) Since a) $\Rightarrow$ c), each conn. cmpt. of G has one fewer edge than vertex. Therefore,

$$\underbrace{|E(G)|}_{n-1} = \sum_{\substack{\text{conn.}\\ \text{cmpts.}}} E(\text{conn. cmpt.}) = \underbrace{|V(G)|}_{n} - \underbrace{\#\text{ components of } G}_{\text{must be } 1}.$$

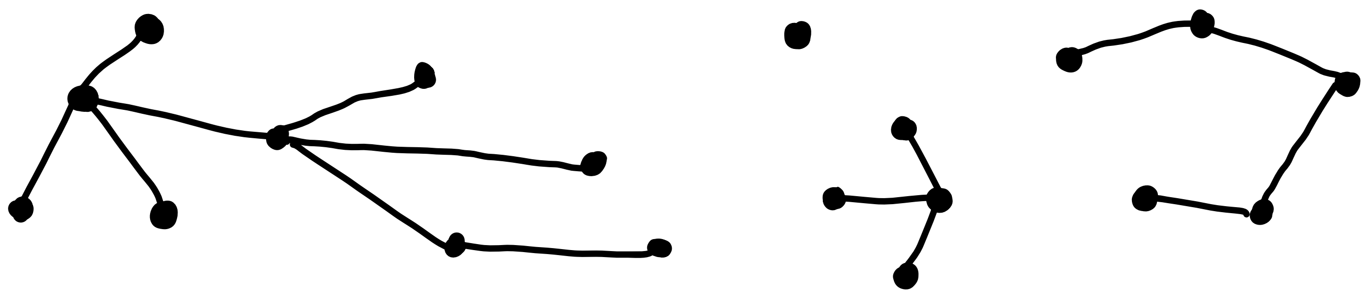
$\square$

Def:

a) A <u>tree</u> is a graph satisfying any/all of the equivalent conditions a) - e) above.



b) A <u>forest</u> is a graph whose conn. components are trees.



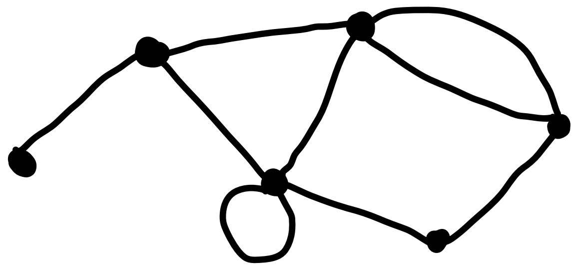
c) A <u>leaf</u> is a degree 1 vertex of a tree

d) $H \subseteq G$ is a <u>spanning subgraph</u> of G if $V(H) = V(G)$ (H, G: any graphs)

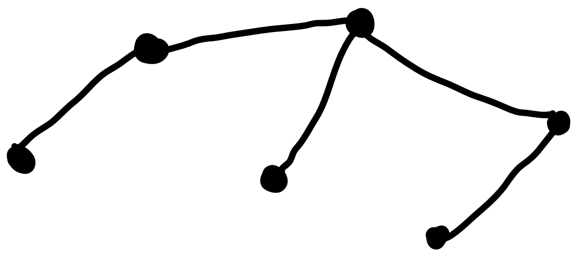
e) Spanning tree $\Leftrightarrow$ spanning subgraph & tree

Class activity: Spanning Subgraph? Spanning Tree?

G



a)



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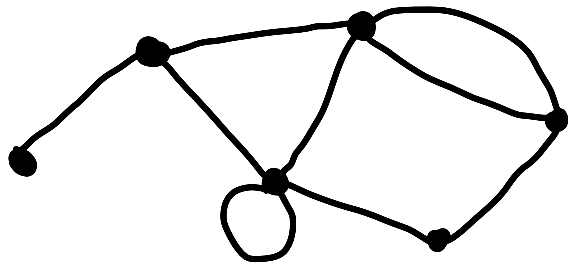
b)



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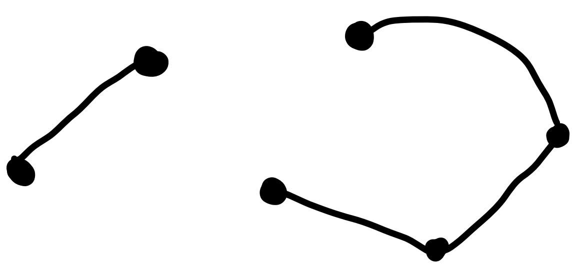
c)



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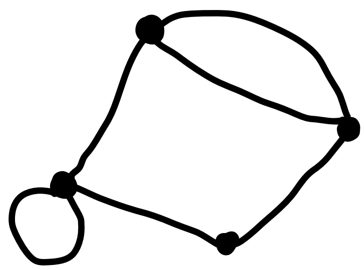
d)



Neither

Cor 2.1.5: Every connected graph G contains a spanning tree.

Pf: Use def'n from Thm 2.1.4e: a tree is a conn. graph s.t. every edge is a cut-edge. If G is a tree, done. Otherwise, G has a non-cut-edge $e \in E(G)$, so $G' := G \setminus e$ is a conn. spanning graph of G w/ fewer edges. So apply induction / extremality.

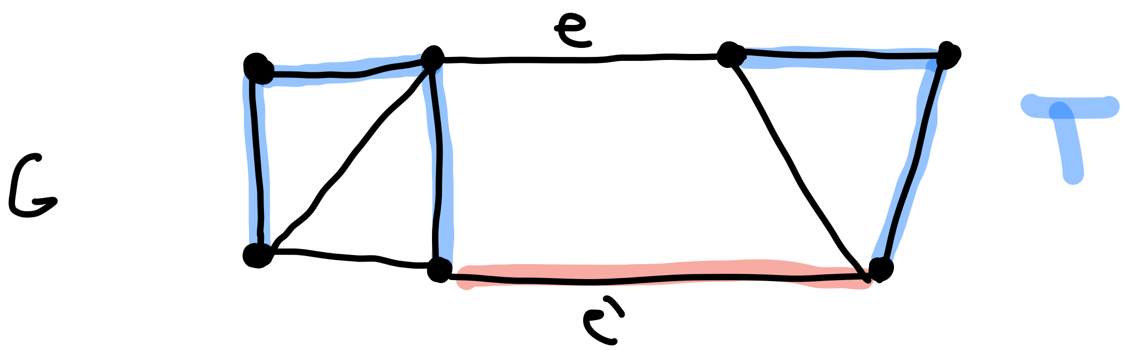
$\square$

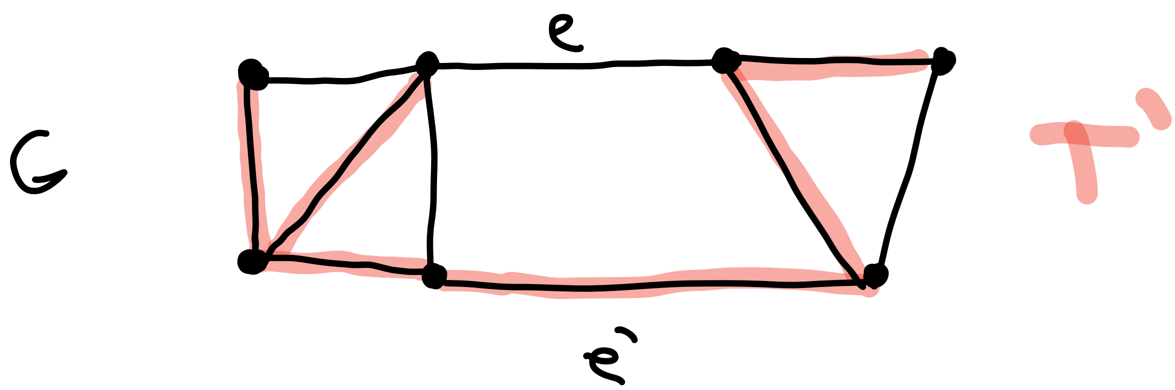
Prop (2.1.6/2.1.7): Let G be a graph w/ spanning trees T, T'.

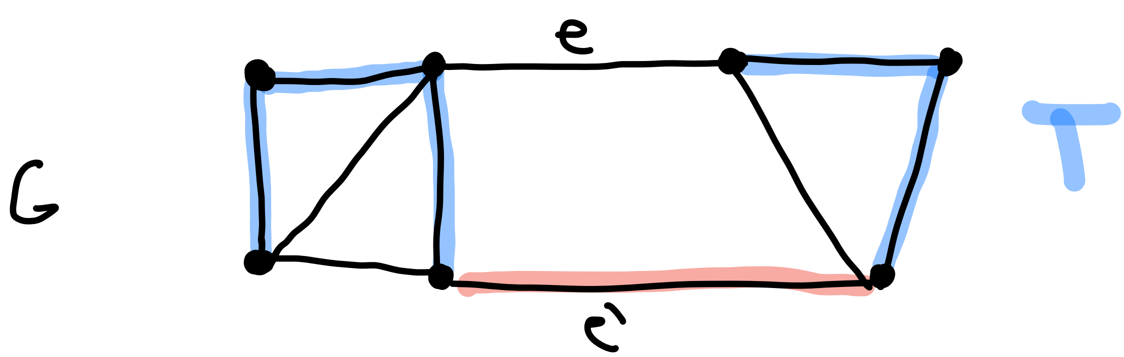
a) For all $e \in E(T)$, $\exists\, e' \in E(T')$ s.t. $(T \cup e') \setminus e$ is a spanning tree of G.

b) For all $e' \in E(T')$, $\exists\, e \in E(T)$ s.t. $(T \cup e') \setminus e$ is a spanning tree of G.

If you tell me which edge to remove, I'll tell you which edge to add
If you tell me which edge to add, I'll tell you which edge to remove


G
e
e'
T


G
e
e'
T'


G
e
e'
T