Note: the distribution of these problems may not match the distribution of exam topics.

Problem §6.3 - 22(d,e,f): How many permutations of the letters $ABCDEFGH$ contain

(d) the strings AB, DE, and GH?

(e) the strings CAB and BED?

(f) the strings BCA and ABF?

Solution. The trick is to treat the strings specified in each part as "superletters" and then to count arrangements of the set of superletters and remaining ordinary letters.

(d) Here, we have three superletters: AB, DE, and GH. This leaves two remaining ordinary letters: C and F. So we're simply counting the number of ways to permute five items, which is by definition $P(5,5) = 5! = 120$.

(e) On first glance, it might seem like we're arranging two superletters (CAB and BED) and three ordinary letters (F, G, and H). But because the letter B appears in both superletters, the only way for a string to contain both CAB and BED is for it to contain the longer substring $CABED$. As such, we're really permuting four objects: $CABED$, F, G, and H. By definition, we can do so in $P(4,4) = 4! = 24$ ways.

(f) Notice that the letters A and B appear in both the string BCA and ABF. In BCA, the letter B is followed by C. But in ABF, it is followed by F. Because it's impossible for B to be followed by *both* C and F in a single string, there are *no* permutations of $ABCDEFGH$ that contain both BCA and ABF as substrings.

□

Problem §6.4 - 20: Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

which relates terms in Pascal's triangle that form a hexagon.

Solution. This problem is really just asking you to get your hands dirty with the definition of a binomial coefficient and some algebra. Diving right in, we can compute

$$\begin{aligned}\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{(n+1)!}{k!(n+1-k)!} \\ &= \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n!}{(n-k+1)!(k-1)!} \cdot \frac{(n+1)!}{(k+1)!(n-k)!} \\ &= \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}\end{aligned}$$

where all we really did was rearrange factorials to get the desired binomial coefficients. □

Problem §6.5 - 16(a,d): How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where x_i, for $i = 1, 2, 3, 4, 5, 6$, is a non-negative integer such that

(a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?

(d) $x_1 < 8$ and $x_2 > 8$?

Solution. Like in lecture, we can think about this using the sticks and stones (or stars and bars) model. Recall that we can write 29 as a summation $1 + \cdots + 1$ of 29 "1"s. We can think about placing sticks between these "1"s to separate them into six "cells"s that represent the x_i. Then, this is really asking us to count the number of ways we can arrange 29 stones and 5 sticks (where the stones represent the copies of "1" and the sticks represent divisions between the x_i), subject to various conditions.

(a) If we require that at least $x_i \geq 2$, this "uses up" 12 of the 29 copies of "1". So this reduces to the problem of counting the number of solutions to the equation

$$x_1' + x_2' + x_3' + x_4' + x_5' + x_6' = 29 - 12 = 17$$

where each $x_i' \geq 0$. We know that this is given by

$$\left(\!\binom{6}{17}\!\right) = \left(\!\binom{6+17-1}{7}\!\right) = \binom{22}{17} = 26,334.$$

(d) We can start by counting the number of solutions that meet the second condition, $x_2 \geq 9$. Similarly to (a), this condition "uses up" 9 of the 29 copies of "1", so this reduces to counting the number of solutions to the equation

$$x_1' + x_2' + x_3' + x_4' + x_5' + x_6' = 20$$

where each $x_i' \geq 0$. We know that this is

$$\left(\!\binom{6}{20}\!\right) = \binom{6+20-1}{20} = \binom{25}{20} = 53,130.$$

But this overcounts, because some of these solutions don't meet the first condition. To count the number of solutions that violate this condition, i.e. where $x_1 \geq 8$, we can observe that in these solutions an additional eight copies of "1" are used up, so we're really counting solutions to the equation

$$x_1'' + x_2'' + x_3'' + x_4'' + x_5'' + x_6'' = 12$$

where each $x_i'' \geq 0$. This is given by

$$\left(\!\binom{6}{12}\!\right) = \binom{6+12-1}{12} = \binom{17}{12} = 6,188.$$

Hence,

$$\begin{aligned}\left(\begin{array}{c}\text{number of solutions with}\\ x_1 < 8 \text{ and } x_2 > 8\end{array}\right) &= \left(\begin{array}{c}\text{number of solutions}\\ \text{with } x_2 > 8\end{array}\right) - \left(\begin{array}{c}\text{number of solutions with}\\ x_2 > 8 \text{ and } x_1 \geq 8\end{array}\right)\\ &= \left(\!\binom{6}{20}\!\right) - \left(\!\binom{6}{12}\!\right)\\ &= 53,130 - 6,188\\ &= 46,942\end{aligned}$$

□

Problem §6.5 - 30: How many different strings can be made from the letters in $MISSISSIPPI$, using all the letters?

Solution. The word $MISSISSIPPI$ contains 11 letters - one "M", four "I"s, four "S"s, and two "P"s. Hence, we can apply the theorem for counting arrangements of a set of objects where some objects are identical. Doing so, we find that we can make

$$\frac{11!}{1!4!4!2!} = 34,650$$

distinct strings. □

Problem §7.1 - 18: What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds?

Solution. We want to compute the probability of the event E that a hand contains five cards of the same suit of *consecutive kinds* (note that now E contains less outcomes than in the previous question, since we're excluding hands with cards that are all of the same suit but not consecutive kinds). To create a straight flush, we can choose the suit of the cards in $\binom{4}{1} = 4$ ways. Because the cards have to be consecutive, the hand is then fully determined once we choose the lowest card in the hand. This lowest card could anything from an ace up through a 9. Hence, there are $\binom{10}{1} = 10$ ways to choose the lowest card in the hand and therefore

$$p(E) = \frac{|E|}{|S|} = \frac{4 \cdot 10}{\binom{52}{5}} = \frac{4 \cdot 10 \cdot 5! \cdot 47!}{52!} = \frac{1}{64974} \approx 0.0015\%$$

□

Problem §7.1 - 24(a): Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding 30.

Solution. Here, the sample space S is the set of all possible ways to choose six integers (without repetition) from the set $\{1, 2, \ldots, 30\}$. There are by definition $\binom{30}{6}$ ways to do so. We want to find the probability of the event E that we choose all six correct numbers. Since there is only one set of six numbers that's correct, clearly $|E| = 1$. Hence,

$$p(E) = \frac{|E|}{|S|} = \frac{1}{\binom{30}{6}} = \frac{1}{593,775}$$

□

Problem §7.2 - 5: A pair of dice is loaded. The probability that a 4 appears on the first die is 2/7, and the probability that a 3 appears on the second die is 2/7. Other outcomes for each die appear with probability 1/7. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

Solution. There are 6 ways to get a total of 7: $(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$. The probability for each is $\frac{1}{7} \cdot \frac{1}{7} = \frac{1}{49}$, except for $(4,3)$, which is $\frac{2}{7} \cdot \frac{2}{7} = \frac{4}{49}$. So in total, we have

$$p(E) = \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \cdot \frac{1}{7} + \frac{2}{7} \cdot \frac{2}{7} + \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \cdot \frac{1}{7} = \frac{9}{49} = 0.1837$$

□

Problem §7.3 - 11: An electronics company is planning to introduce a new camera phone. The company commissions a marketing report for each new product that predicts either the success or the failure of the product. Of new products introduced by the company, 60% have been successes. Furthermore, 70% of their successful products were predicted to be successes, while 40% of failed products were predicted to be successes. Find the probability that this new camera phone will be successful if its success has been predicted.

Solution. We use Bayes' Theorem. Let E be the prediction of success, and F be actual success. Then, $p(F) = 0.6$, $p(E|F) = 0.7$, $p(E|\overline{F}) = 0.4$, and

$$p(F|E) = \frac{p(E|F)p(F)}{p(E)} = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})} = \frac{0.7 \cdot 0.6}{0.7 \cdot 0.6 + 0.4 \cdot 0.4} = 0.7241.$$

□

Problem §8.1 - 8:

(a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

(b) What are the initial conditions?

(c) How many bit strings of length seven contain three consecutive 0s?

Solution. This is very similar to the example we did in Lecture 24, where we found a recurrence relation for the number of binary strings with a pair of consecutive 0s. The only difference is that now we will need to go back one term further in the sequence and therefore will have a recurrence relation whose order is higher by one.

(a) Let a_n denote the number of binary strings of length n that contain three consecutive 0s. Just like in lecture, we'll think about counting the number of such strings by breaking the way we could construct such a string into a few cases. Such strings could:

1. Start with a 1, followed by a string of length $n-1$ containing three consecutive 0s.
2. Start with 01, followed by a string of length $n-2$ containing three consecutive 0s.
3. Start with 001, followed by a string of length $n-3$ containing three consecutive 0s.
4. Start with 000, followed by any binary stringh of length $n-3$.

These four cases are mutually exclusive and together contain every possible binary string of length n with three consecutive 0s. Note that this method of construction is only valid for $n \geq 3$. By counting the number of strings in each case, we can write down the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3} \qquad \text{for } n \geq 3.$$

(b) We can then determine the initial conditions, i.e. the values of a_0, a_1, and a_2. Since it is not possible to have three consecutive 0s in a binary string with length less than three, we have

$$a_0 = a_1 = a_2 = 0.$$

(c) Finally, we're asked to compute a_7. To do so, we can simply use repeated applications of the recurrence relation:

$$\begin{aligned}
a_3 &= a_2 + a_1 + a_0 + 2^0 = 0 + 0 + 0 + 1 = 1 \\
a_4 &= a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3 \\
a_5 &= a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8 \\
a_6 &= a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20 \\
a_7 &= a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47
\end{aligned}$$

□

Problem §8.2 - 26(a,c): What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

(a) $F(n) = n^2$?

(c) $F(n) = n2^n$?

Solution. In order to apply Theorem 6, we need to know the characteristic roots of the associated linear homogeneous recurrence relation, $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$. To determine the characteristic roots, we let $a_n = r^n$ and substitute into the associated linear homogeneous recurrence relation.

$$r^n = 6r^{n-1} - 12r^{n-2} + 8r^{n-3}$$
$$r^3 = 6r^2 - 12r + 8$$

We then solve for r to find

$$r^3 - 6r^2 + 12r - 8 = 0$$
$$(r-2)^3 = 0$$

from which we see that the associated linear homogeneous solution has a single characteristic root, $r_0 = 2$, with multiplicity 3.

Now, we can deal with finding the particular solutions for the given functions.

(a) Recall that we could instead write $F(n)$ in the form $F(n) = n^2 \cdot 1^n$. Because 1 is not a characteristic root of the associated homogeneous recurrence relation, we know from Theorem 6 that the particular solution has the form

$$a_n^{(p)} = p_2n^2 + p_1n + p_0,$$

where p_2, p_1, and p_0 are constants.

(c) Because 2 is a characteristic root with multiplicity 3, we know from Theorem 6 that the particular solution has the form

$$a_n^{(p)} = (p_1n + p_0) \cdot n^3 \cdot 2^n$$

□

Problem §8.5 - 24: Find the probability that when a fair coin is flipped five times tails comes up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads.

Solution. Let S be the sample space, i.e. the set of all possible outcomes when flipping a fair coin five times. Let E_1 be the event that tails comes up exactly three times, E_2 be the event that the first and last flips come up tails, and E_3 be the event that the second and fourth flips come up heads. The problem is then asking us to compute $p(E_1 \cup E_2 \cup E_3) = |E_1 \cup E_2 \cup E_3|/|S|$.

Clearly, $|S| = 2^5$ since there are exactly two outcomes for each flip (heads or tails). To find $|E_1 \cup E_2 \cup E_3|$, we can apply the principle of inclusion-exclusion. We know that

$$|E_1 \cup E_2 \cup E_3| = (|E_1| + |E_2| + |E_3|) - (|E_1 \cap E_2| + |E_2 \cap E_3| + |E_1 \cap E_3|) - |E_1 \cap E_2 \cap E_3|$$

Observe that $|E_1| = \binom{5}{3}$, i.e. the number of ways to choose which three of the five flips come up tails. To see that $|E_2| = 2^3$, observe tat the first and last flips are required to be tails and the middle three flips could be either heads or tails. Similarly, $|E_3 = 2^3|$ because we're again counting the number of possible outcomes when two of the flips are fixed.

Next, consider $|E_1 \cap E_2|$. Since two flips are known to be tails, the number of outcomes in $|E_1 \cap E_2|$ is the number of ways to choose the third flip that comes up tails. Hence, $|E_1 \cap E_2| = \binom{3}{1} = 3$. Next, observe that $|E_1 \cap E_3| = 1$. because only one outcome meets the criteria to belong to both events - flipping $THTHT$. Similarly, $|E_2 \cap E_3| = 2$ because there are exactly two outcomes that meet the criteria for both events -$THTHT$ and $THHHT$. Finally, $|E_1 \cap E_2 \cap E_3| = 1$ because only the outcome $THTHT$ meets the criteria to belong to all three events.

Plugging these values into our formula, we compute

$$\begin{aligned}|E_1 \cup E_2 \cup E_3| &= (|E_1| + |E_2| + |E_3|) - (|E_1 \cap E_2| + |E_2 \cap E_3| + |E_1 \cap E_3|) - |E_1 \cap E_2 \cap E_3| \\ &= \left(\binom{5}{3} + 2^3 + 2^3\right) - (3 + 1 + 2) + 1 \\ &= 21\end{aligned}$$

and therefore

$$p(E_1 \cup E_2 \cup E_3) = \frac{|E_1 \cup E_2 \cup E_3|}{|S|} = \frac{21}{32}.$$

□

Problem §8.6 - 16: A group of n students is assigned seats for each of two classes in the same classroom. How many ways can these seats be assigned if no student is assigned the same seat for both classes?

Solution. This is a derangement problem. When students choose seats in their first class, which they can do in $n!$ ways, the numbers $1, \ldots, n$ are assigned to the chairs based on the student who sits there (i.e., the chair where student i sits in the first class is labeled $'i'$). We then want to count the number of ways that the chairs can be reassigned for the second class such that student i is not sitting in chair i, for all $1 \leq i \leq n$. We recognize this as the number of derangements of n objects, i.e. D_n. Hence,

$$\begin{aligned}\begin{pmatrix}\text{number of seat assignments such that no} \\ \text{student is assigned the same seat in both classes}\end{pmatrix} &= n! \cdot D_n \\ &= (n!)^2 \cdot \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \cdot \frac{1}{n!}\right]\end{aligned}$$

□

Problem §9.1 - 42: List the 16 different relations on the set $A = \{0, 1\}$.

Solution. $A \times A = \{(0,0), (0,1), (1,0), (1,1)\}$, so the 16 relations are the 16 possible subsets of this set: $\emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}, \{(0,0),(0,1)\}, \{(0,0),(1,0)\}, \{(0,0),(1,1)\}, \{(0,1),(1,0)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \{(0,0),(0,1),(1,0)\}, \{(0,0),(0,1),(1,1)\}, \{(0,0),(1,0),(1,1)\}, \{(0,1),(1,0),(1,1)\}, \{(0,0),(0,1),(1,0),(1,1)\}$, □

Problem §9.1 - 44(a,c,d,f): Which of the 16 relations on $A = \{0, 1\}$, which you listed in Exercise 42, are reflexive? Symmetric? Antisymmetric? Transitive?

Solution. $\emptyset$: symmetric, antisymmetric, transitive.
$\{(0,0)\}$: symmetric, antisymmetric, transitive.
$\{(0,1)\}$: antisymmetric, transitive.
$\{(1,0)\}$: antisymmetric, transitive.
$\{(1,1)\}$: symmetric, antisymmetric, transitive.
$\{(0,0),(0,1)\}$: antisymmetric, transitive.

$\{(0,0),(1,0)\}$: antisymmetric, transitive.
$\{(0,0),(1,1)\}$: reflexive, symmetric, antisymmetric, transitive.
$\{(0,1),(1,0)\}$: symmetric.
$\{(0,1),(1,1)\}$: antisymmetric, transitive.
$\{(1,0),(1,1)\}$: antisymmetric, transitive.
$\{(0,0),(0,1),(1,0)\}$: symmetric.
$\{(0,0),(0,1),(1,1)\}$: reflexive, antisymmetric, transitive.
$\{(0,0),(1,0),(1,1)\}$: reflexive, antisymmetric, transitive.
$\{(0,1),(1,0),(1,1)\}$: symmetric.
$\{(0,0),(0,1),(1,0),(1,1)\}$: reflexive, symmetric, transitive.

□

Problem §9.3 - 6: How can the matrix representing a relation R on a set A be used to determine whether the relation is asymmetric?

Solution. R is asymmetric if and only if $m_{ij} + m_{ji} \leq 1$ for all i and j i.e. M_R has all zeroes on the diagonal, and for every off-diagonal entry m_{ij}, m_{ij} and m_{ji} *aren't* both 1. □

Problem §9.3 - 13: Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrices representing $R^{-1}, \overline{R}, R \circ R$.

Solution.

$$M_{R^{-1}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = M_R.$$

$$M_{\overline{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$M_{R \circ R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

□

Problem §6.2: 31: Show that there are at least six people in California (population: 37 million) with the same three initials who were born on the same day of the year (but not necessarily the same year). Assume that everyone has three initials.

Solution. Here, the "pigeons" are the people living in California and the "pigeonholes" are all the possible pairs of initials and days of the year. In order to apply the pigeonhole principle, we need to count the number of pigeonholes. Because there are 26 ways to choose each of the 3 initials and 366 ways to choose the birthday, the product rule tells us that there are $26^3 \cdot 366 = 6,432,816$ possible combinations. The generalized pigeonhole principle then tells us that there must be at least

$$\left\lceil \frac{37,000,000}{6,432,816} \right\rceil = 6$$

people with the same combination. □

Problem §6.2: 40: Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Solution. Suppose there are $n \geq 2$ people at the party. Assume that "knowing" is symmetric, so either two people mutually know each other or they don't. Let K be a function that maps each person x at the party to the number of people that they know, $K(x)$. We would like to apply the pigeonhole principle, where the "pigeons" are the people at the party and the "pigeonholes" are the number of people that they know at the party.

On the face of it, the possible values for $K(x)$ are $0, 1, \ldots, n-1$ (i.e., ranging from knowing nobody at the party to knowing everyone). However, it's not possible for both 0 and $n-1$ to appear in the range of K, because if one person at the party knows nobody then it's not possible for another person to know everyone. Hence, the range of K contains $n-1$ elements. Because there are n people at the party, we have more "pigeons" than "pigeonholes" and therefore the pigeonhole principle guarantees that there are at least two people at the party who know the same number of other people there. $\square$