# Solutions to Math 213-A1 Midterm Exam 3 — Apr. 29, 2026

1. (16 points) Answer the following questions.
   *(No work necessary for this problem! Only your answer will be graded.)*

   (a) (4 points) Let $A = \{1, 2, 3, 4, 5, 6\}$, and let $R$ be the equivalence relation corresponding to the set partition $\{\{1, 3, 5\}, \{2, 6\}, \{4\}\}$. How many ordered pairs are in $R$?

   Each set inside the set partition contributes $k^2$ ordered pairs (where $k$ is the size of the set). Therefore,
   $$|R| = 3^2 + 2^2 + 1^2 = 9 + 4 + 1 = 14.$$

   (b) (4 points) A full 3-ary rooted tree has 40 total vertices. How many leaves does it have?

   A full $m$-ary tree with $n$ total vertices has $\frac{(m-1)n+1}{m}$ leaves. Setting $m = 3$ and $n = 40$, we get a total of $\frac{2\cdot 40+1}{3} = \frac{81}{3} = 27$ leaves. Alternately, notice that $40 = 1 + 3 + 9 + 27$, and these are the vertices at each level of a full 3-ary tree of height 3.

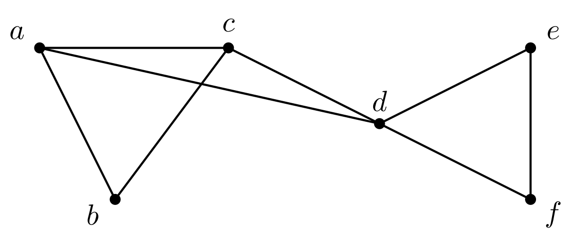
   (c) (4 points) True or false: $K_{3,4}$ is planar.

   False. $K_{3,4}$ contains $K_{3,3}$ as a subgraph, and $K_{3,3}$ is nonplanar.
   (Alternatively: $K_{3,4}$ is bipartite, so it has no triangles. By Corollary 1 from §10.7, any simple planar graph with no triangles satisfies $e \le 2v - 4$. Since $K_{3,4}$ has $v = 7$ and $e = 12$, we would need $12 \le 2(7) - 4 = 10$, which is false.)

   (d) (4 points) What is the chromatic number of the cycle graph $C_7$?

   $\chi(C_7) = 3$, since $C_7$ is an odd cycle. Any odd cycle requires 3 colors since it cannot be 2-colored.

2. (14 points) Consider the following graph $G$.




   (a) (6 points) Determine whether $G$ has an Eulerian circuit, an Eulerian path, or neither. *(You are not required to find the circuit/path if it exists, although you may)*

   The degrees are: $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 3$, $\deg(d) = 4$, $\deg(e) = 2$, $\deg(f) = 2$. The vertices $a$ and $c$ have odd degree, and these are the only two vertices with odd degree. Therefore, $G$ has an Eulerian path (starting and ending at $a$ and $c$) but not an Eulerian circuit.
   One such path is $a, b, c, a, d, e, f, d, c$.

   (b) (6 points) Determine whether $G$ has a Hamiltonian circuit, a Hamiltonian path, or neither. *(You are not required to find the circuit/path if it exists, although you may)*

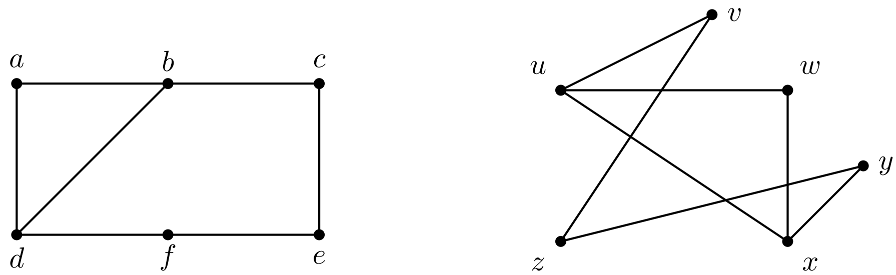
   $G$ has a Hamiltonian path but not a Hamiltonian circuit.
   *Hamiltonian path:* $a, b, c, d, e, f$ is a Hamiltonian path (edges $ab, bc, cd, de, ef$ are all in $G$).
   *No Hamiltonian circuit:* Since $\deg(e) = 2$, both edges incident to $e$ (which are $de$ and $ef$) must be used in any Hamiltonian circuit. Similarly, since $\deg(f) = 2$, both edges incident to $f$ (which are $df$ and $ef$) must be used. Thus the circuit contains edges $de$, $ef$, and $df$. But then the two edges of the circuit at $d$ are $de$ and $df$, so no other edge at $d$ (i.e., $ad$ or $cd$) is in the circuit. This means no edge of the circuit connects $\{a, b, c\}$ to $\{d, e, f\}$, so the circuit cannot visit all vertices.

(c) (2 points) What are the cut vertices of $G$? *(No work needed for this part)*

The only cut vertex is $d$. Removing $d$ disconnects the graph into $\{a, b, c\}$ and $\{e, f\}$.

3. (16 points) Determine if the given pair of graphs is isomorphic. Either exhibit an isomorphism or provide a rigorous argument that none exists.

(a) (8 points)

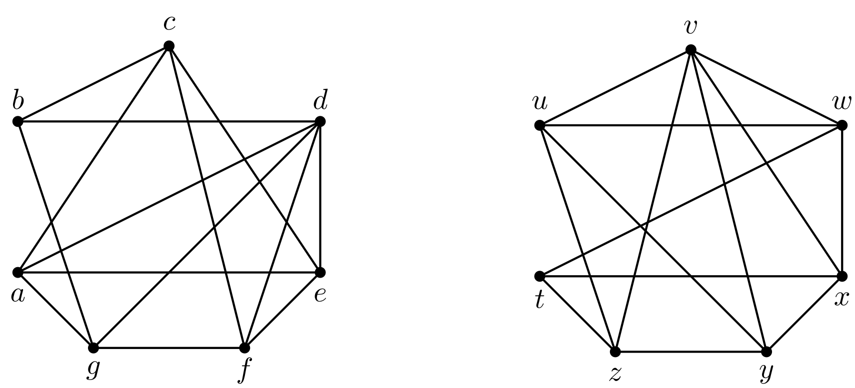



These graphs are isomorphic. Both have 6 vertices, 7 edges, and degree sequence $3, 3, 2, 2, 2, 2$. Let $f$ be the function $p(a) = w, p(b) = u, p(c) = v, p(d) = x, p(e) = z, p(f) = y$. Then $p$ is a bijection. We verify that adjacency is preserved:

$$ab \leftrightarrow wu, \quad ad \leftrightarrow wx, \quad bc \leftrightarrow uv, \quad bd \leftrightarrow ux,$$
$$ce \leftrightarrow vz, \quad df \leftrightarrow xy, \quad ef \leftrightarrow zy.$$

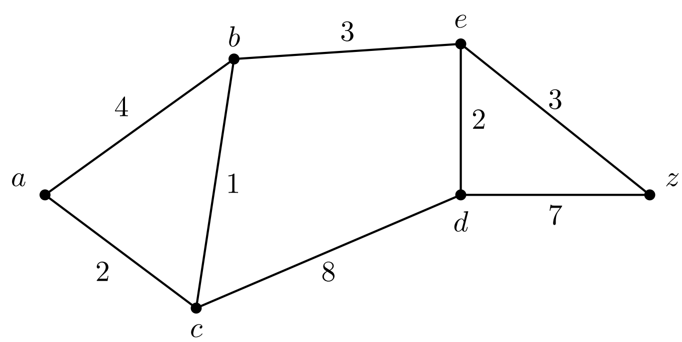
Each pair on the left is an edge of the left graph, and each pair on the right is an edge of the right graph, and these account for all 7 edges of each graph. Therefore, $p$ is a graph isomorphism.

(b) (8 points)




These graphs are not isomorphic, despite having the same degree sequences. To see that they aren't isomorphic, note that the left graph has exactly one vertex of degree 3 ($b$), and exactly one vertex of degree 5 ($d$), and these vertices are adjacent. On the other hand, the right graph has exactly one vertex of degree 3 ($t$), and exactly one vertex of degree 5 ($v$), but these vertices are *not* adjacent. Therefore, the graphs are not isomorphic since any isomorphism $p$ must preserve degrees, and so we would have $p(b) = t$ and $p(d) = v$, but $b$ and $d$ are adjacent while $t$ and $v$ aren't.

4. (8 points) Find the length of a shortest path between $a$ and $z$ in the following weighted graph. **You must use Dijkstra's algorithm.** In each step, record the set $S$ of vertices that have been processed and the current distance label $L(v)$ for every vertex $v$.




We use Dijkstra's algorithm, starting from $a$.

**Start:** $S = \emptyset$.
$L(a) = 0, \quad L(b) = \infty, \quad L(c) = \infty, \quad L(d) = \infty, \quad L(e) = \infty, \quad L(z) = \infty.$

**Step 1:** Add $a$ to $S$. $S = \{a\}$.
Update neighbors of $a$: $L(b) = \min(\infty, 0 + 4) = 4$; $L(c) = \min(\infty, 0 + 2) = 2$.
$L(a) = 0, \quad L(b) = 4, \quad L(c) = 2, \quad L(d) = \infty, \quad L(e) = \infty, \quad L(z) = \infty.$

**Step 2:** Smallest label outside $S$ is $L(c) = 2$. Add $c$. $S = \{a, c\}$.
Update: $L(b) = \min(4, 2 + 1) = 3$; $L(d) = \min(\infty, 2 + 8) = 10$.
$L(a) = 0, \quad L(b) = 3, \quad L(c) = 2, \quad L(d) = 10, \quad L(e) = \infty, \quad L(z) = \infty.$

**Step 3:** Smallest outside $S$ is $L(b) = 3$. Add $b$. $S = \{a, b, c\}$.
Update: $L(e) = \min(\infty, 3 + 3) = 6$.
$L(a) = 0, \quad L(b) = 3, \quad L(c) = 2, \quad L(d) = 10, \quad L(e) = 6, \quad L(z) = \infty.$

**Step 4:** Smallest outside $S$ is $L(e) = 6$. Add $e$. $S = \{a, b, c, e\}$.
Update: $L(d) = \min(10, 6 + 2) = 8$; $L(z) = \min(\infty, 6 + 3) = 9$.
$L(a) = 0, \quad L(b) = 3, \quad L(c) = 2, \quad L(d) = 8, \quad L(e) = 6, \quad L(z) = 9.$

**Step 5:** Smallest outside $S$ is $L(d) = 8$. Add $d$. $S = \{a, b, c, d, e\}$.
Update: $L(z) = \min(9, 8 + 7) = 9$. (No change.)
$L(a) = 0, \quad L(b) = 3, \quad L(c) = 2, \quad L(d) = 8, \quad L(e) = 6, \quad L(z) = 9.$

**Step 6:** Add $z$. $S = \{a, b, c, d, e, z\}$. Done.

The length of the shortest path from $a$ to $z$ is $\boxed{9}$.

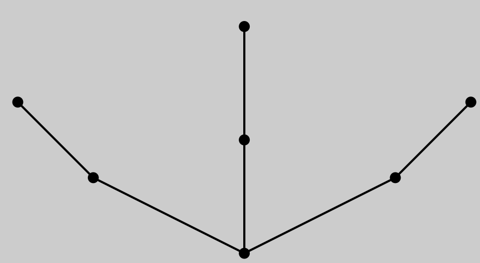
(For reference, the shortest path itself is $a, c, b, e, z$ with length $2 + 1 + 3 + 3 = 9$.)

5. (15 points) For each of the following, either draw a graph with the given properties, or explain why no such graph can exist.

   (a) (5 points) A tree with 7 vertices and degree sequence $3, 2, 2, 2, 1, 1, 1$.

   Such a tree exists. The sum of the degrees is $3 + 2 + 2 + 2 + 1 + 1 + 1 = 12$, so the graph has $12/2 = 6$ edges. A tree with 7 vertices has exactly $7 - 1 = 6$ edges, so this is consistent.

   Take a center vertex $v_1$ of degree 3 adjacent to $v_2, v_3, v_4$, and then attach one additional leaf to each:



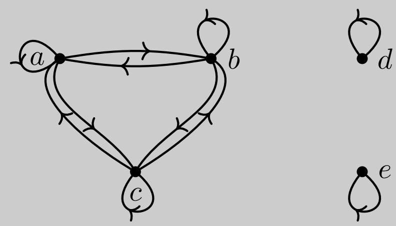
This tree has degree sequence $3, 2, 2, 2, 1, 1, 1$ as required.

(b) (5 points) A simple planar graph with chromatic number 6.

Such a graph cannot exist. By the four-color theorem, the chromatic number of any planar graph is at most 4.

(c) (5 points) A digraph with exactly 11 edges which represents an equivalence relation on the set $A = \{a, b, c, d, e\}$.

The equivalence relation must have 11 ordered pairs. Since every equivalence relation on $A$ must contain the 5 pairs $(a, a), (b, b), (c, c), (d, d), (e, e)$, the remaining 6 pairs must come from elements in the same equivalence class. One partition that works is $\{\{a, b, c\}, \{d\}, \{e\}\}$, which gives $3^2 + 1^2 + 1^2 = 11$ pairs. The corresponding digraph is:




6. (10 points) Suppose that $G$ is a connected planar simple graph with $v \geq 5$ vertices, $e$ edges, and no simple circuits of length less than 5. Prove that

$$e \leq \frac{5(v-2)}{3}.$$

*(Hint: apply the same proof technique as Corollary 1 in §10.7, which we proved in class)*

Consider a planar representation of $G$, and let $r$ denote the number of regions. By Euler's formula,

$$v - e + r = 2. \qquad (*)$$

Since every simple circuit in $G$ has length at least 5, the boundary of each region traverses at least 5 edges, so the degree of each region is at least 5. Since the sum of the degrees of all regions equals $2e$ (each edge borders exactly two regions), we obtain

$$2e = \sum_{\text{regions } R} \deg(R) \geq 5r,$$

so $r \leq \frac{2e}{5}$.

Substituting into $(*)$:

$$2 = v - e + r \leq v - e + \frac{2e}{5} = v - \frac{3e}{5}.$$

Rearranging: $\frac{3e}{5} \leq v - 2$, and therefore

$$e \leq \frac{5(v-2)}{3}.$$