Solutions to Math 213-A1 Midterm Exam 2 — Apr. 1, 2026

1. (28 points) Answer the following questions.
(No work necessary for this problem! Only your answer will be graded.)
(For this problem, you may leave your answer in terms of binomial coefficients.)

(a) (4 points) If 35 pigeons are placed in 8 holes, what is the largest number k such that there are *guaranteed* to be at last k pigeons in at least one hole?

By the generalized pigeonhole principle, $k = \lceil 35/8 \rceil = 5$.

(b) (4 points) A weighted coin has a 60% probability that a given flip is heads. If the coin is flipped 3 times, what is the probability that there are exactly two heads?

By the formula for Bernoulli Trials, the probability is $\binom{3}{2}(0.6)^2 \cdot (0.4)^1 = 3 \cdot (0.6)^2 \cdot (0.4)^1$

(c) (4 points) How many ways are there to pack 4 identical copies of a book into at most 3 indistinguishable boxes?

This is just the number of ways to write 4 as a sum of positive integers in decreasing order, with at most 3 terms: 4; $3+1$; $2+2$; $2+1+1$. Total: 4 ways
(Note that $1+1+1+1$ doesn't work since it has 4 terms)

(d) (4 points) Let $A = \{a, b, c, d\}$ and let R be the following relation on A:

$$R = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}.$$

Is R reflexive? Symmetric? Antisymmetric? Transitive? (Give your answer to all four of these questions).

R is antisymmetric and transitive, but not reflexive or symmetric.

(e) (4 points) How many ways are there to pack 10 distinguishable objects into 4 distinguishable boxes?

Repeated product rule gives 4^{10}.

(f) (4 points) How many ways are there to buy 9 cookies from a shop with 3 flavors if you can buy as many as you want of each flavor.

Sticks-and-stones: $\binom{9+3-1}{9} = \binom{11}{9}$.

(g) (4 points) What is the probability that a 5-card poker hand is two pairs (i.e. contains two of one kind, two of a second kind, and one of a third kind).

We choose two of the 13 kinds to have a pair, then one of the remaining 11 kinds to have the remaining card. For each of the pairs, we choose two of the four cards of that kind, and for the fifth card, we choose one of the four cards. Finally, we divide by the total number of hands, to get:

$$\frac{\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}} = \frac{123552}{2598960} = 0.0475.$$

2. (15 points) Find all solutions to the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2} + 2^n$

This is a linear inhomogeneous recurrence relations, so all solutions are of the form $a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(p)}$ is any particular solution and $a_n^{(h)}$ is a solution to the homogeneous recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2}.$$

The characteristic equation is $r^2 - 2r - 3 = (r-3)(r+1) = 0$, so there are two roots, 3 and -1. Therefore, (by Theorem 8.2.3) the general solution to the homogeneous equation is

$$a_n^{(h)} = \alpha \cdot 3^n + \beta \cdot (-1)^n,$$

where α and β are arbitrary.

Next, for the particular solution. Since the inhomogeneous part is $F(n) = 2^n$ and 2 is not a root of the characteristic equation, we know (Theorem 8.2.6) that there is a particular solution of the form $a_n^{(p)} = p2^n$, for some (but not all!) values of p.

Finally, we plug this particular solution into the recurrence relation to obtain

$$p2^n = 2p2^{n-1} + 3p2^{n-2} + 2^n.$$

Factoring out 2^{n-2}, this equation becomes $4p = 4p + 3p + 4$, and solving for p we get $p = -4/3$.

Therefore, the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha \cdot 3^n + \beta \cdot (-1)^n - \frac{4}{3} \cdot 2^n,$$

for arbitrary α and β.

3. (10 points) Suppose that the words "April 1st" appear in 20% of news articles that turn out to be pranks and in 1% of news articles that turn out to be legitimate. If 10% of news articles are pranks, what is the probability that an article containing "April 1st" is a prank?

We use Bayes' Theorem. Given an article, let E be the event that it is a prank, and let F be the event that it contains the words "April 1st". We want to find $p(E|F)$

From the problem statement, $p(E) = 0.1, p(\overline{E}) = 0.9$, $p(F|E) = 0.2$, and $p(F|\overline{E}) = 0.01$.

Applying Bayes Theorem, we have

$$p(E|F) = \frac{p(F|E)p(E)}{p(F)} = \frac{p(F|E)p(E)}{p(F|E)p(E) + p(F|\overline{E})p(\overline{E})} = \frac{0.2 \cdot 0.1}{0.2 \cdot 0.1 + 0.01 \cdot 0.9} = 0.6897 \approx 69\%.$$

(Since no calculators are allowed, $\frac{0.2 \cdot 0.1}{0.2 \cdot 0.1 + 0.01 \cdot 0.9}$ is an acceptable answer)

4. (10 points) Let A, B, and C be sets with $|A| = 7$, $|B| = 6$, $|C| = 9$, $|A \cap B| = 3$, $|A \cap C| = 3$, $|B \cap C| = 4$, and $|A \cup B \cup C| = 14$. Find $|A \cap B \cap C|$.

By inclusion-exclusion, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Solving for $|A \cap B \cap C|$, we have

$$\begin{aligned} |A \cap B \cap C| &= |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| \\ &= 14 - 7 - 6 - 9 + 3 + 3 + 4 \\ &= 2 \end{aligned}$$

5. (15 points) Let n and j be positive integers. **Using a combinatorial argument**, prove the following

identity:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{j} = \binom{n}{j} 2^{n-j}.$$

We count the number of ways to choose, out of a group of n people, a committee (of any size), and out of that committee, a smaller "executive committee." (Equivalently, given a set A with $|A| = n$, we choose a nested pair of subsets $S \subseteq T \subseteq A$ where $|S| = j$). We do this in two ways.

On one hand, we can choose the executive committee first. There are $\binom{n}{j}$ ways to do this. Then with the remaining $n - j$ people, we choose whether or not they are on the larger committee. There are two ways (yes or no) to make this choice for each person, so 2^{n-j} ways in total. Thus there are $\binom{n}{j} 2^{n-j}$ ways to make this sequence of choices.

On the other hand, let k be the size of the larger committee. Fixing k, we can choose the k people to be on the larger committee, and there are $\binom{n}{k}$ ways to make this choice. Then out of these k people we can choose the j people to be on the executive committee, and there are $\binom{k}{j}$ ways to do this step. Summing over k, we have $\sum_{k=0}^{n} \binom{n}{k}\binom{k}{j}$ ways to make this sequence of choices.

Since both approaches count the number of ways to make the same decision, we have

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{j} = \binom{n}{j} 2^{n-j},$$

as desired.