Solutions to Math 213-A1 Midterm Exam 1 — Feb. 18, 2026

1. (15 points) Prove that for all positive integers n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let $P(n)$ be the statement

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We prove that $P(n)$ is true for all n by induction on n. Base case: If $n = 1$, then

$$1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6},$$

so $P(1)$ is true.

Induction step: Assume that $P(k)$ is true for some $k \geq 1$. We prove that $P(k+1)$ is true.

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6},
\end{aligned}$$

where the first equality uses the inductive hypothesis. Thus, $P(k+1)$ is true, and therefore $P(n)$ is true for all n by induction.

2. (24 points) Answer the following questions. *(No work necessary for this problem!)*

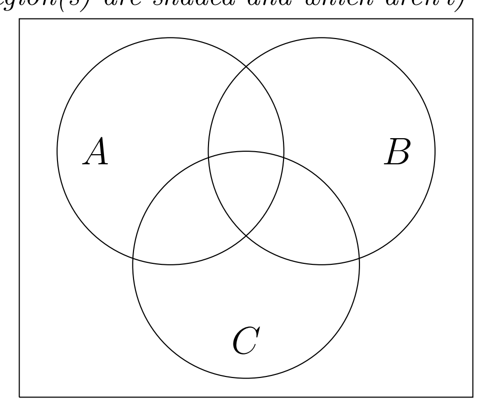
   (a) (4 points) Let P be the statement "There is an exam today" and Q be the statement "It is Wednesday". Express the proposition $(\neg Q) \implies (\neg P)$ as an English sentence.

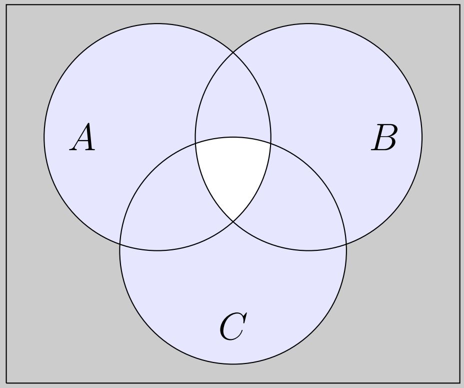
   If it is not Wednesday, then there is no exam today.

   (b) (4 points) Let A, B, and C be sets. Shade the Venn diagram below to represent the set

$$(A \setminus B) \cup (B \setminus C) \cup (C \setminus A).$$

   (Be very clear about which region(s) are shaded and which aren't)





(c) (4 points) List three of the seven properties that algorithms must have (according to Rosen).

Input, output, finiteness, correctness, definiteness, effectiveness, generality

(d) (4 points) What is the smallest integer n such that $f(x) = \frac{x^5 \log x - 3x^2 + 1/x}{5x^3 - 3 \sin x}$ is $O(x^n)$?

$n = 3$

(e) (4 points) How many one-to-one functions are there from A to B if $|A| = 3$ and $|B| = 5$?

$5 \cdot 4 \cdot 3 = 60$

(f) (4 points) How many 7-digit codes are there consisting only of the digits $1, 2, 3$, and 4? (e.g. 1111111, 1312443)

There are 4 choices for each of the 7 digits, so by the product rule, there are 4^7 codes in total.

3. (10 points) Prove that $f(x) = 2x^3 + (\log x)^2 + 7$ is $O(x^3)$.

(*Note: for this problem, you can assume that* $\log x < x$ *for all positive* x*)*

Let $k = 1, C = 10$. Then for $x > k$, $(\log x)^2 < x^2 < x^3$ and $7 < 7x^3$, so

$$\begin{aligned} |f(x)| &= 2x^3 + (\log x)^2 + 7 \\ &< 2x^3 + x^3 + 7x^3 \\ &= 10x^3 \\ &= C|x^3|, \end{aligned}$$

so $f(x)$ is $O(x^3)$.

4. (12 points) Determine, with justification, whether each of these functions is injective, surjective, bijective, or none.

(a) (6 points) $f : A \to B$, where

$$A = \{a, b, c, d\}, \quad B = \{x, y, z\}, \qquad f(a) = x, \quad f(b) = y, \quad f(c) = y, \quad f(d) = z.$$

f is surjective, but not injective. It is surjective because $f(a) = x, f(b) = y, f(d) = z$, so all elements of B are in the image of f. It is not injective because $f(b) = y = f(c)$, but $b \neq c$.

(b) (6 points) $f : C \to \mathbb{Z}$, where C is the set of all even integers, and $f(n) = n/2$.

f is bijective. It is injective because if $f(x) = f(y)$, then $x/2 = y/2$, and multiplying both sides by 2 gives $x = y$. It is surjective because if n is in $\mathbb{Z}$, then $2n$ is an even integer, so it is in C, and $f(2n) = n$, so n is in the image of f.

5. (25 points) (a) (10 points) Prove for any sets A and B that

$$\mathcal{P}(A \setminus B) \setminus \{\emptyset\} \subseteq \mathcal{P}(A) \setminus \mathcal{P}(B).$$

($A \setminus B$ can equivalently be written $A - B$)

Let $S \in \mathcal{P}(A \setminus B)$. By the definition of the power set, $S \subseteq A \setminus B$. This means that every element of S is an element of A but not an element of B. Since every element of S is an element of A, this means that $S \subseteq A$, so $S \in \mathcal{P}(A)$. On the other hand, since no element of S is an element of B, the only way that $S \subset B$ is if $S = \emptyset$. But if $S \in \mathcal{P}(A \setminus B) \setminus \{\emptyset\}$, then $S \neq \emptyset$, so we have $S \not\subseteq B$, so $S \notin \mathcal{P}(B)$. Since $S \in \mathcal{P}(A)$ and $S \notin \mathcal{P}(B)$, we have that $S \in \mathcal{P}(A) \setminus \mathcal{P}(B)$.

(b) (5 points) For the sets $A = \{1, 2, 3\}, B = \{3, 4, 5\}$, what are the sets $\mathcal{P}(A \setminus B) \setminus \{\emptyset\}$ and $\mathcal{P}(A) \setminus \mathcal{P}(B)$? (*No work needed for this part*)

We have

$$\mathcal{P}(A \setminus B) \setminus \{\emptyset\} = \mathcal{P}(\{1, 2\}) \setminus \{\emptyset\} = \{\{1\}, \{2\}, \{1, 2\}\}$$

and

$$\mathcal{P}(A) \setminus \mathcal{P}(B) = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

(c) (10 points) Suppose that $|A| = m$, $|B| = n$, and $|A \cap B| = p$. Let $C = \mathcal{P}(A \setminus B) \setminus \{\emptyset\}$ and let $D = \mathcal{P}(A) \setminus \mathcal{P}(B)$. We know from part (a) than $C \subseteq D$. Determine the cardinality of $D \setminus C$ in terms of m, n, and p.

Since $C \subseteq D$, we have $|D \setminus C| = |D| - |C|$, since we are removing every element of C from D to obtain $D \setminus C$.

Now, since $|A| = m, |B| = n$, and $|A \cap B| = p$, we have $|A \setminus B| = m - p$, so $|\mathcal{P}(A \setminus B)| = 2^{m-p}$, and $|C| = |\mathcal{P}(A \setminus B) \setminus \emptyset| = 2^{m-p} - 1$.

On the other hand, $|\mathcal{P}(A)| = 2^m$, and while $|\mathcal{P}(B)| = 2^n$, not every subset of B is in $\mathcal{P}(A)$. In fact, exactly those subsets of B which are also contained in A are in $\mathcal{P}(A)$, this is 2^p of them. So $|D| = 2^m - 2^p$.

Overall, we have

$$|D \setminus C| = |D| - |C| = (2^m - 2^p) - (2^{m-p} - 1) = 2^m - 2^p - 2^{m-p} + 1 = (2^{m-p} - 1)(2^p - 1).$$

Another approach would be to notice that any element of D that is not in C is a subset of A that contains at least one element in B and at least one element not in B. Such an element S is a union of some nonempty subset of B and some nonempty subset of $A \setminus B$. There are $2^p - 1$ nonempty subsets of B and $2^{m-p} - 1$ nonempty subsets of $A \setminus B$, so by the product rule, there are a total of $(2^{m-p} - 1)(2^p - 1)$ ways to choose S.