Announcement

Quiz 2: this Wednesday (2/11)

Recall: Principle of Mathematical Induction:

$P(n)$ is true for all n if and only if

- $P(1)$ is true (base case)
- If we assume $P(k)$ is true (for arbitrary k), then $P(k+1)$ is true (induction step)

Def: The statement $P(k)$ in the inductive step is called the <u>inductive hypothesis</u> (since we assume it's true)

Remark: The textbook has more on the history/philosophy of induction

Ex 2: Find and prove a formula for the sum of the first n odd integers $1+3+\dots+(2n-1)$

$n=1$: 1

$n=2$: $1+3=4$

$n=3$: $1+3+5=9$

$n=4$: $1+3+5+7=16$

Let $P(n)$ be the statement:

$$1+3+\cdots+(2n-1)=n^2$$

We prove $P(n)$ for all n by induction

Base case: $1=1^2$, so $P(1)$ is true.

Inductive step: Assume that $P(k)$ is true. Then,

$$\begin{aligned} 1+3+\cdots+(2k-1)+(2k+1) &= k^2+(2k+1) \quad \text{(by the inductive hypothesis } P(k)\text{)} \\ &= k^2+2k+1 \\ &= (k+1)^2 \end{aligned}$$

so $P(k+1)$ is true, and so $P(n)$ is true for all n by induction.

$\square$

Ex 6: Prove that 2^n is $O(n!)$.

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120

Pf: Let $k=4$, $C=1$. We prove that 2^n is $O(n!)$ by showing that $|2^n| < C|n!|$ for all $n > k$.

Let $P(n)$ be the statement $2^n < n!$. We want to prove that $P(n)$ is true for all $n \geq 5$. We prove this by induction.

Base case: $n=5$ (note: modified starting point!)

When $n=5$, $2^n = 32 < 120 = n!$, so $P(5)$ is true.

Inductive step: Suppose that $P(a)$ is true, and $a \geq 5$. We want to show $P(a+1)$ is true. We have,

$$2^{a+1} = 2 \cdot 2^a < 2 \cdot a! < (a+1)a! = (a+1)!,$$

so $P(a+1)$ is true. Therefore, $P(n)$ is true for all $n \geq 5$ by induction, so 2^n is $O(n!)$. $\square$

Ex 8: Prove that $\underbrace{n^3-n \text{ is divisible by } 3}_{3 \mid n^3-n}$ for all positive integers n.

Pf: Let $P(n)$ be the statement $3 \mid n^3-n$. We prove that $P(n)$ is true for all n by induction.

Base case: If $n=1$, $n^3-n = 1^3-1 = 0 = 0 \cdot 3$, so

$P(1)$ is true.

Inductive step: Assume $P(k)$ is true. Then, $3 \mid k^3 - k$, so let $k^3 - k = 3m$, where m is an integer.

Then,

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= 3m + 3(k^2 + k) \\ &= 3(m + k^2 + k)\end{aligned}$$

So $(k+1)^3 - (k+1)$ is divisible by 3, and $P(k+1)$ is true.

Thus, $P(n)$ is true for all n by induction. $\blacksquare$

§5.2: Strong Induction and Well-Ordering

Strong induction:

$P(n)$ is true for all n if and only if

- $P(1)$ is true (base case)
- If we assume $P(1), P(2), \ldots, P(k)$ are all true, then $P(k+1)$ is true (induction step)

Only difference: now we get to assume $P(1), \ldots, P(k)$ in the induction step instead of just $P(k)$

No downside to using strong induction

Sometimes just assuming $P(k)$ is enough

Other times, it helps to assume $P(1), \dots, P(k)$

Ex 2: Show that if $n \in \mathbb{Z}$, $n \geq 2$, then n can be written as a product of one or more primes

Pf: We prove this by induction. Let $P(n)$ be the statement "n can be written as a product of primes"

Base case: 2 is prime, so $P(2)$ is true

Inductive step: Assume that $P(1), \dots, P(k)$ are true.

If $k+1$ is prime, then $P(k+1)$ is true. Otherwise, $k+1 = ab$ for integers $2 \leq a, b < k+1$. By the inductive hypothesis, a and b can be written as products of primes:

$$a = p_1 p_2 \cdots p_n, \quad b = q_1 q_2 \cdots q_m.$$

Then

$$k+1 = p_1 \cdots p_n q_1 \cdots q_m$$

can also be written as a product of primes, so $P(k+1)$ is true, and $P(n)$ is true for all $n \geq 2$ by induction

$\square$

Ex 4: Prove that every amount of postage of 12 cents or more can be formed using just 4 cent and 5 cent stamps

Pf: We prove this via strong induction. Let $P(n)$ be the statement "n cents can be formed using just 4-cent & 5-cent stamps"

Base cases: $12 = 4 + 4 + 4$

$13 = 4 + 4 + 5$

$14 = 4 + 5 + 5$

$15 = 5 + 5 + 5$

So $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are all true.

Inductive step: Let $k \geq 15$, and assume that $P(12), \dots, P(k)$ are all true. We want to show that $P(k+1)$ is true. Since $k \geq 15$, $k+1 \geq 16$, so $k+1-4 \geq 12$, so $P(k+1-4)$ is true. Therefore, we can make $k+1-4$ cents using 4-cent and 5-cent stamps, so adding a 4-cent stamp gives $k+1$ cents. Thus, $P(k+1)$ is true, so $P(n)$ is true for all $n \geq 12$ by strong induction $\square$

Ex 3: Consider the following game: Two piles of n matches

The players take turns removing ≥ 1 matches from one of the piles. The players who takes the last match wins. Show that Player 2 can always guarantee a win.

Class activity: Play this game, and try to figure out a strategy.

Pf: We use strong induction. Let $P(n)$ be "Player 2 can win whenever there are initially n matches in each pile"

Base case: If $n=1$, Player 1 must remove the 1 match from one of the piles. Player 2 takes the match from the other pile and wins.

Inductive step: Suppose $k \geq 1$ and $P(1), \dots, P(k)$ are true. For $k+1$ matches per pile, suppose Player 1 takes r matches from the first pile. Then Player 2 can take r matches from the other pile. If $r = k+1$, Player 2 wins. If $1 \leq r < k+1$, then each pile has $k+1-r$

matches remaining, and it is Player 1's turn again. Since $1 \leq k+1-r \leq k$, $P(k+1-r)$ is true, so Player 2 can now guarantee a win. Thus, $P(k+1)$ is true, so by strong induction, $P(n)$ is true for all n. $\square$