Recall: Let f, g be functions from $\mathbb{Z}$ or $\mathbb{N}$ or $\mathbb{R}$ to $\mathbb{R}$.

We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$.

If f is $O(g)$, then g is $\Omega(f)$

If f is $O(g)$ and g is $O(f)$, then f is $\Theta(g)$

Ex 3.2.9: Show that $f(x) = (x+1)\log(x^2+1)$ is $\Theta(x \log x)$

Pf: We show that a) $f(x)$ is $O(x \log x)$ and b) $x \log x$ is $O(f(x))$

b) Notice that $\log x$ is an increasing function. Let $k = 1$, $C = 1$. Then if $x > k$,

$$x \log x \leq x \log(x^2) \qquad (\text{since } x^2 \geq x)$$

$$\leq x \log(x^2+1) \qquad (\text{since } x^2+1 \geq x^2)$$

$$\leq (x+1)\log(x^2+1) \qquad (\text{since } x+1 \geq x)$$

$$= C\,|f(x)|,$$

so $x \log x$ is $O(f(x))$.

a) Let $k=3$, $C=6$. Then if $x>k$,

$$f(x) = (x+1)\log(x^2+1) \leq (x+1)\log(2x^2) \qquad (\text{since } x^2 \geq 1)$$

$$= (x+1)(\log 2 + \log x + \log x) \qquad (\text{by log rules})$$

$$\leq (x+1)\, 3 \log x \qquad (\text{since } x>2, \text{ so } \log x > \log 2)$$

$$< (2x) \cdot 3 \log x \qquad (\text{since } x > 1)$$

$$= 6x \log x.$$

Therefore, $f(x) = O(x \log x)$. $\square$

Ex 11: Let $f(n) = 1+2+3+\cdots+n$. Show that f is $\Theta(n^2)$.

Pf: We show that a) f is $O(n^2)$ and b) f is $\Omega(n^2)$.

a) Let $k = C = 1$. Then if $n > k$,

$$
\begin{aligned}
f(n) &= 1+2+\cdots+n \\
&\le n+n+\cdots+n \qquad (\text{since } 1,2,\ldots,n-1 \le n) \\
&= n^2 \\
&= C\,|n^2|,
\end{aligned}
$$

so f is $O(n^2)$.

b) Let $k=1$, $C = 1/4$. Then if $n > k$,

$$
\begin{aligned}
f(n) &= 1+\cdots+\lceil n/2 \rceil + \lceil n/2 \rceil + 1 + \cdots + n \\
&\ge \lceil n/2 \rceil + \lceil n/2 \rceil + 1 + \cdots + n \qquad (\text{throw out first terms}) \\
&\ge \lceil n/2 \rceil + \lceil n/2 \rceil + \cdots + \lceil n/2 \rceil \qquad (\text{since } \lceil n/2 \rceil < \lceil n/2 \rceil + 1, \ldots, n) \\
&\ge n/2 + n/2 + \cdots + n/2 \qquad (\text{since } \lceil n/2 \rceil \ge n/2
\end{aligned}
$$

$$\geq \left(n/2\right)\left(n/2\right) \qquad \left(\text{since there are } \geq n/2 \text{ integers in the range } \lceil n/2 \rceil, \dots, n\right)$$

$$= n^2/4$$

$$= c\,|n^2|$$

Thus, $f(n)$ is $\Omega(n^2)$. $\square$

Scratch work:

$$1 + 2 + \cdots + \underbrace{\lceil n/2 \rceil + \lceil n/2 \rceil + 1 + \cdots + n}$$

$$\geq$$

$$\geq \underbrace{\lceil n/2 \rceil + \lceil n/2 \rceil + \cdots + \lceil n/2 \rceil}_{\geq \frac{n}{2} \text{ terms each of which is } \geq n/2}$$

e.g.

$1+2+\underbrace{3+4+5}$

$1+2+\underbrace{3+4+5+6}$

$$\geq \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)$$

$$= \frac{n^2}{4}$$

§5.1: Mathematical Induction

Ex 1: Show that if n is a positive integer, then

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Let's check a couple of cases:

$n=1$: $1=\frac{1\cdot 2}{2}$ ✓

$n=2$: $1+2=3=\frac{2\cdot 3}{2}$ ✓

$n=3$: $1+2+3=6=\frac{3\cdot 4}{2}$ ✓

Seems like it probably works

In this case, there's a trick:

$$\left.\begin{array}{rl} & 1+2+\cdots+n \\ + & n+n-1+\cdots+1 \end{array}\right\} 2\cdot LHS$$

$$\overline{n+1+n+1+\cdots+n+1} = n(n+1)$$

But in general, we want a better tool

$n=3$: LHS $= 1+2+3$ RHS $= \dfrac{3\cdot 4}{2}$

$n=4$: LHS $= 1+2+3+4$ RHS $= \dfrac{4\cdot 5}{2}$

bigger by 4

bigger by

$$\frac{4\cdot 5}{2} - \frac{3\cdot 4}{2} = \frac{4}{2}(5-3) = 2\cdot 2 = 4$$

What about for general n?

Assume that

$$1+2+\cdots+n = \frac{n(n+1)}{2} \quad (*)$$

WTS:

$$1+2+\cdots+n+n+1 = \frac{(n+1)(n+2)}{2}$$

$$1+2+\cdots+n+n+1 = \frac{n(n+1)}{2} + n+1 \qquad (\text{by } *)$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2} \quad \checkmark$$

So if the equation holds for $n=1$ AND whenever it holds for n it holds for $n+1$, it <u>must</u> hold for all n.

Let $P(n)$ be a statement (true or false) depending on the positive integer n.

Want to show that $P(n)$ is true for all n

Principle of Mathematical Induction:

$P(n)$ is true for all n if and only if

- $P(1)$ is true (base case)
- If we assume $P(k)$ is true (for arbitrary k), then $P(k+1)$ is true (induction step)

Ex 1 (cont).

Pf: Let $P(n)$ be the statement

$$1 + \cdots + n = \frac{n(n+1)}{2}$$

We prove $P(n)$ is true for all n by induction on n.

Base case: When $n=1$,

$$1 = \frac{1 \cdot 2}{2}, \text{ so } P(1) \text{ is true.}$$

Inductive step: Assume that $P(k)$ is true. Then,

$$1 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + k+1 \qquad (\text{by } P(k))$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2},$$

so $P(k+1)$ is true. Therefore, $P(n)$ is true for all n by induction. $\square$

Def: The statement $P(k)$ in the inductive step is called the <u>inductive hypothesis</u> (since we assume it's true)

Remark: The textbook has more on the history/philosophy of induction

Ex 2: Find and prove a formula for the sum of the first n odd integers $1+3+\dots+(2n-1)$

$n=1$: 1

$n=2$: $1+3=4$

$n=3$: $1+3+5=9$

$n=4$: $1+3+5+7=16$

Let $P(n)$ be the statement:

$$1 + 3 + \cdots + (2n-1) = n^2$$

We prove $P(n)$ for all n by induction

Base case: $1 = 1^2$, so $P(1)$ is true.

Inductive step: Assume that $P(k)$ is true. Then,

$$\begin{aligned} 1 + 3 + \cdots + (2k-1) + (2k+1) &= k^2 + (2k+1) \quad \text{(by the inductive hypothesis } P(k)\text{)} \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

So $P(k+1)$ is true, and so $P(n)$ is true for all n by induction.

$\square$