Announcements

HW3 posted (due Wed. 2/11 @ 9am)

Exam 1 will be Wed. 2/18 in class
(2 weeks from today)

Sorting algorithms:

General problem: Sort a list in increasing order

Many different algorithms

Bubble sort algorithm:

Input: list of integers $a_1, \dots, a_n$

Output: list of integers which is the original list in inc. order

Algorithm:

for $i := 1$ to $n-1$
  for $j := 1$ to $n-i$
    if $a_j > a_{j+1}$
      Swap a_j and a_{j+1}
return $a_1, \dots, a_n$

Class activity: perform bubble sort on the list

3, 2, 4, 1, 5

Insertion sort algorithm (if time):

Input: list of integers $a_1, \ldots, a_n$

Output: list of integers which is the original list in inc. order

Algorithm:

for $j := 2$ to n

    Let $i := 1$

    While $a_j > a_i$ (finding the spot for a_j)

        $i := i+1$

    Let $m := a_j$ (insert a_j into spot m)

    for $k := 0$ to $j-i-1$

        $a_{j-k} := a_{j-k-1}$ (shift other elts. to make room)

    $a_i =: m$

return $a_1, \ldots, a_n$

Class activity: perform insertion sort on the list

$3, 2, 4, 1, 5$

§3.2: The growth of functions

Def (big-O notation): Let f, g be functions from $\mathbb{Z}$ or $\mathbb{R}$ to $\mathbb{R}$. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$.

Idea: $f(x)$ is $O(g(x))$ if "eventually", $g(x)$ is bigger. If they are "the same" up to a constant, then f is $O(g)$ and g is $O(f)$.

Def: If f is $O(g)$, then g is $\Omega(f)$ "Omega"

If f is $O(g)$ and g is $O(f)$, then f is $\Theta(g)$ "Theta"

Think:

$$\left.\begin{aligned} f \text{ is } O(g) &\leftrightarrow f \leq g \\ f \text{ is } \Omega(g) &\leftrightarrow f \geq g \\ f \text{ is } \Theta(g) &\leftrightarrow f = g \end{aligned}\right\}$$

this is <u>not</u> literally true, just a good way to think of it

Examples:

a) x^2 is $O(x^3)$, so x^3 is $\Omega(x^2)$

b) $3x^2+17x+6$ is $O(x^{2.1})$

c) $3x^2+17x+6$ is $O(x^2)$ and x^2 is $O(3x^2+17x+6)$

so $3x^2+17x+6$ is $\Theta(x^2)$

d) x^a is $O(e^x)$ for any a

e) $n!$ is $\Omega(e^n)$

n	e^n	$n!$
1	e	1
2	e^2	$1\cdot 2$
3	e^3	$1\cdot 2\cdot 3$
4	e^4	$1\cdot 2\cdot 3\cdot 4$
5	e^5	$1\cdot 2\cdot 3\cdot 4\cdot 5$

f) x^a is $\Omega(1)$ if $a \geq 0$

x^a is $O(1)$ if $a \leq 0$

g) $\log x$ is $O(x)$

Simple tricks:

1) Larger powers grow faster
2) Ignore constant factors
3) Only worry about the fastest-growing term

Now for some proofs:

Ex 1: Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$

Pf: We need to find $C, k \in \mathbb{R}$ s.t.

$$|f(x)| \leq C|x^2| \text{ whenever } x > k$$

Let $k = 10$, $C = 5$. Then if $x > k$,

$$\begin{aligned} f(x) &= x^2 + 2x + 1 \\ &< x^2 + 2x^2 + x^2 \qquad \text{(since } x \geq k \geq 1) \\ &= 4x^2 \\ &< 5x^2 \\ &= C|x^2|. \end{aligned}$$

Therefore, $f(x)$ is $O(x^2)$. $\square$

Note: we chose C and k much bigger than needed. $k = 1, C = 4$ or $k = 3, C = 2$ would have worked.

Ex 9: Show that $f(x) = (x+1)\log(x^2+1)$ is $\Theta(x \log x)$.

Pf: We show that a) $f(x)$ is $O(x \log x)$ and b) $x \log x$ is $O(f(x))$.

b) Notice that $\log x$ is an increasing function. Let $k = 1$, $C = 1$. Then if $x > k$,

$$\begin{aligned} x \log x &\leq x \log(x^2) && (\text{since } x^2 \geq x) \\ &\leq x \log(x^2+1) && (\text{since } x^2+1 > x^2) \\ &\leq (x+1)\log(x^2+1) && (\text{since } x+1 > x) \\ &= C|f(x)| \end{aligned}$$

a) Let $k =$, $C =$. Then if $x > k$,

$$\begin{aligned} (x+1)\log(x^2+1) &\leq (x+1)\log(2x^2) && (\text{since } 1 \leq x^2) \\ &= (x+1)(\log 2 + \log x + \log x) && (\text{by log rules}) \\ &\leq (x+1)\, 3 \log x && (\text{since } x > 2, \text{ so } \log x > \log 2) \\ &< 2x \cdot 3 \log x && (\text{since } x > 1) \\ &= 6x \log x \end{aligned}$$

$\square$