

Announcements

Final exam: Tuesday, 5/12, 8:00am-11:00am, Everitt 3117

Covers entire course

Two reference sheets allowed (see policy email)

Review session: Mon. 5/11, time and location TBD

Office hours: TBD

Practice problems posted

Final exam review

Partial list of topics:

Logic

Propositions

English vs. symbols

and, or, not, implies, for all, there exists

Truth tables

Sets

Roster notation vs. set builder notation

Special sets (\mathbb{Z} , \mathbb{R} , \emptyset , etc.)

Venn diagrams

Subset, power set, Cartesian product

Cardinality

Set operations: union, intersection, set-minus, complement

Set identities (1-10)

Proof techniques: elt. chasing, membership tables

Functions

Definition

Domain, codomain, range/image, preimage

Injective/surjective/bijective (& pf. techniques)

Composition

Inverses + invertibility

Algorithms

Definition

Properties (describe and check)

Perform an algorithm

Write an algorithm

Searching/sorting/greedy change

Big-O

Precise def'n of O , Ω , Θ ; proof techniques

Tricks & heuristics ($1 < \log x < x < x^2 < \dots < e^x < \dots$)

Induction

Mathematical vs. strong

Base case, inductive step

Critique proofs

Various examples from class & H/W

Counting

Sum/product/subtraction/division rules + combining the rules
(Generalized) pigeonhole principle

Permutations/combinations, and generalized versions

Binomial coeffs., identities, and the binom. thm

Probability

Def'n's (event, sample space, etc.)

Basic examples (e.g. coins, dice, cards)

Independence

Bernoulli trials

Conditional probability & Bayes' Thm.

Recurrence rel'n's

Basic ideas, examples

Linear (in)homogeneous rec. rel'n's, and how to solve e.g. Thm. 6

Inclusion - Exclusion & applications (integer eq'ns, derangements)

Relations

Properties: reflexive, symmetric, antisymmetric, transitive

Operations: complement, inverse, composition

Representing relations: matrix, digraph

Equivalence rel's

Definition

Equiv. classes and set partitions

Graphs/digraphs

Def'n's: simple/multi./nbhd./deg./bipartite/(induced) subgraph

Handshake thm.

Special classes of graphs

Constructions: deletion/contraction/union

Adjacency & incidence matrices

Isomorphism

Show that graphs are isomorphic: explicit isom., adj. matrices

Show that graphs are not isomorphic: different "label-indep. properties"

Connectivity (for digraphs, weak vs. strong), cut-edges/cut-vertices

Paths/circuits

Eulerian/Hamiltonian (+ criteria for Eulerian)

Shortest path problems

Weighted graphs

Dijkstra's algorithm

Travelling salesperson

Planar graphs

Direct pf. of planarity/nonplanarity

Regions, degree, etc.

Euler's formula and consequences

Graph coloring

Maps vs. graphs and their colorings

Chromatic number

Four-color theorem

Trees

Definitions

Properties

Rooted trees, m -ary trees

Applications: binary search trees, decision trees, game trees

Spanning trees, minimal spanning trees, Prim's & Kruskal's algorithms

Examples:

1) Let $A = \{a, b, c, d\}$ Does there exist an equivalence relation on A that is

a) antisymmetric

b) not antisymmetric

c) asymmetric

d) the assoc. digraph is conn., but has no simple circuits of length 3.

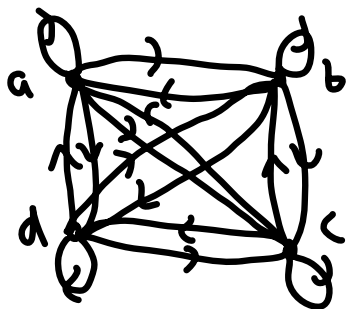
Sol'n:

a) Yes, $R = \{(a, a), (b, b), (c, c), (d, d)\}$

b) Yes e.g. $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$

c) No. Asymmetric means antisym and $(x, x) \notin R$ for all $x \in A$, so (for nonempty A), no relation can be both reflexive and asymmetric

d) No. If the assoc. digraph is conn., it is the complete digraph:



and a, b, c, d, a is one of the many simple circuits of length 3.

2) When rolling three dice, what is the conditional probability that the product is at least 10 given that the sum is 7

Sol'n: Possible ways to roll a sum of 7:

511 (3 orders) prod. is 5

421 (6 orders) prod. is 8

331 (3 orders) prod. is 9

322 (3 orders) prod. is 12

Number of ways to roll a sum of 7: 15

Num. of these ways where the prod. is ≥ 10 : 3

Conditional prob.: $\frac{3}{15}$

3) Prove that $f(n) = n^2 e^n$ is $O(n^n)$

Pf: Let $C=1, k=10$. Then for $n > k$,

$$\frac{n}{e} > \frac{10}{e} > \frac{10}{3} > 3,$$

$$\text{so } \frac{n^n}{e^n} = \left(\frac{n}{e}\right)^n > 3^n.$$

Next, we show that if $n \geq 3$, $n^2 < 3^n$ by induction.

Base case: If $n=3$, $n^2=9 < 27=3^3$.

Inductive step: Suppose that $n \geq 3$ and $n^2 < 3^n$.

Then $\frac{n+1}{n} \leq \frac{4}{3} < 1.5$, so

$$(n+1)^2 = n^2 \cdot \left(\frac{n+1}{n}\right)^2 < n^2 \cdot (1.5)^2 = 2.25n^2 < 2.25 \cdot 3^n < 3^{n+1}.$$

Thus, we have shown that if $n \geq 3$, $n^2 < 3^n$ by induction.

Therefore, if $n > k$,

$$|f(n)| = n^2 e^n < n^2 \frac{n^n}{3^n} < n^n,$$

so f is $O(n^n)$. □

4) Determine whether $f: \mathbb{Z} \rightarrow \mathbb{Z}$ is ^(injective) one-to-one, ^(surjective) onto, both, or neither.

(bijective)

a) $f(n) = \lfloor n/2 \rfloor$

b) $f(n) = n + (-1)^n$

c) $f(n) = 3n - 2$

Soln:

a) Not one-to-one since $f(0) = f(1) = 0$

Onto since if $y \in \mathbb{Z}$, $f(2y) = y$

b) One-to-one since $f(n) = n \pm 1$, so n and $f(n)$ always have the opposite parity

Let $x, y \in \mathbb{Z}$, $x \neq y$

- If x even, y odd, $f(x)$ is odd, $f(y)$ is even, so $f(x) \neq f(y)$
- If x odd, y even, $f(x)$ is even, $f(y)$ is odd, so $f(x) \neq f(y)$
- If x even, y even, $f(x) = x+1 \neq y+1 = f(y)$
- If x odd, y odd, $f(x) = x-1 \neq y-1 = f(y)$

Onto since if $z \in \mathbb{Z}$, let

$$x = \begin{cases} z+1, & \text{if } z \text{ even} \\ z-1, & \text{if } z \text{ odd} \end{cases}$$

- If z even, x is odd, so $f(x) = x-1 = z$
- If z odd, x is even, so $f(x) = x+1 = z$

Alternate method: f is bijective because it is invertible.

(Recall: g is the inverse of f if $g \circ f = f \circ g = \text{id}$)

• If x even, $(f \circ f)(x) = f(f(x)) = f(x+1) = x$

• If x odd, $(f \circ f)(x) = f(f(x)) = f(x-1) = x$

So $f^{-1} = f$ (!) and so f is invertible and therefore bijective (don't have to check both orders since $f^{-1} = f$)

c) One-to-one since if $3x-2 = 3y-2$, $3x = 3y$, so $x = y$

Not onto. $f(n) = 3n-2$ is always 2 less than a multiple of 3 i.e. the remainder when dividing $f(n)$ by 3 is always 1 i.e. $f(n)$ is always in the equiv. class $[1]$ in the equiv. rel'n $a \sim b$ if $a-b$ is a mult. of 3 (congruence class)

So in particular $0 \notin \text{range}(f)$, so f is not onto.

5) a) Find a recurrence rel'n for the number of ways to pay a toll of $\$d$ dollars, using $\$1$ and $\$5$ bills.

b) In how many different ways can a driver pay a toll of $\$17$?

Sol'n: a) $a_n = a_{n-5} + a_{n-1}$ for $n \geq 5$

b) The initial conditions are

$$a_0 = a_1 = a_2 = a_3 = a_4 = 1.$$

We can compute $a_5 = 2, a_6 = 2, \dots, a_{17} = 4$