Announcements

Midterm 3: Wed 4/29 in class

Covers through end of this week

Reference sheet allowed (one A4 sheet, both sides)

Practice problem sol'ns posted

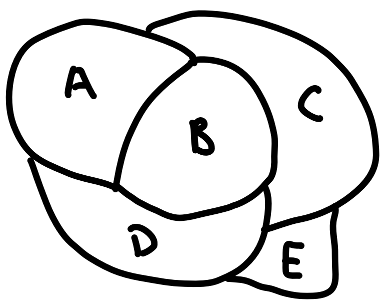
See policy email

HW 11 posted (due Wed. 5/6)

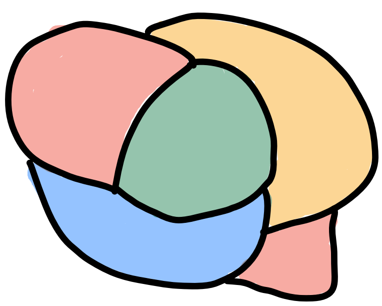
Shall we cancel an upcoming class?

§10.8: Graph coloring

Map: Separation of (part of) the plane into contiguous regions



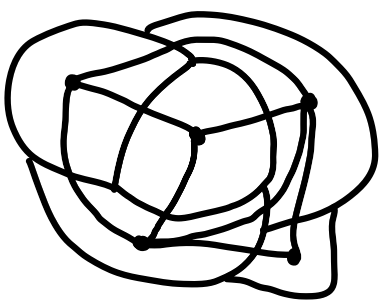
Map coloring: assign each region a color s.t. all adjacent regions have different colors

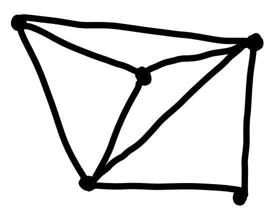


Question: what is the smallest number of colors we need for a given map?

This is secretly a graph theory problem

Def: For a map M, the <u>dual graph</u> of M is the graph formed by putting a vertex in the middle of each region of M, and connecting vertices for adjacent regions





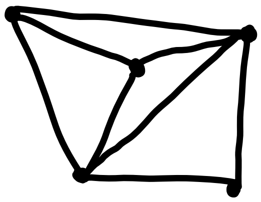
Graph coloring: assign each vertex a color s.t. all adjacent vertices have different colors

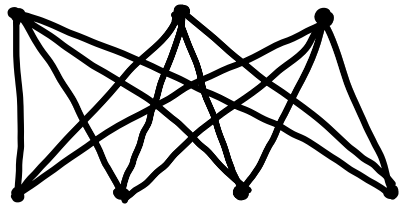
(coloring the dual graph of a map is equiv. to coloring the map itself)

Def: The <u>chromatic number</u> $\chi(G)$ "chi" is the smallest number of colors needed to color G.

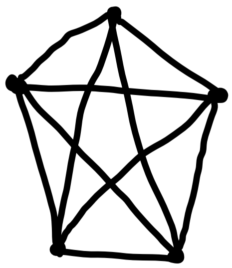
We can do this for any graph; the planar ones are the graphs corresponding to maps.

Class activity: Find $\chi(G)$

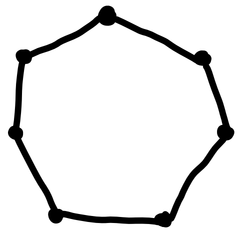




$K_{3,4}$



K_5



C_7

Examples:

$\chi(K_n) = n$

$\chi(K_{m,n}) = 2$

$$\chi(C_n) = \begin{cases} 2, & \text{if } n \text{ even} \\ 3, & \text{if } n \text{ odd} \end{cases}$$

$$\chi(W_n) = \begin{cases} 3, & \text{if } n \text{ even} \\ 4, & \text{if } n \text{ odd} \end{cases}$$

$\chi(Q_n) = 2$

Claim: $\chi(G) \leq 2 \iff G$ is bipartite

Pf: $G = (V, E)$ is bipartite if and only if V can be written $V = V_1 \sqcup V_2$ where there are no edges

disjoint union

in E w/ both endpoints in V_1 or both endpoints in V_2.

Color the vertices in V_1 red and the vertices in V_2 blue; this is a 2 coloring of G, so $\chi(G) \leq 2$.

Conversely, if $\chi(G) \leq 2$, then there exists a coloring of G using (at most) red and blue. Let V_1 be the red vertices and V_2 be the blue vertices. Then $V = V_1 \sqcup V_2$ and there are no edges w/ both endpoints in V_1 or both endpoints in V_2. $\square$

Four-color theorem: For any simple planar graph G, $\chi(G) \le 4$

1852: Conjectured by Guthrie

1879: ~~Proof given by Kempe~~

1890: Heawood showed that Kempe's proof was flawed (!)
and also proved the five-color thm.

⋮

many years passed

⋮

1976: Appel & Haken proved the theorem!
@UIUC

Proof technique: break up into 1,834 cases,
and check them all by computer

To this day, no non-computer proof exists!