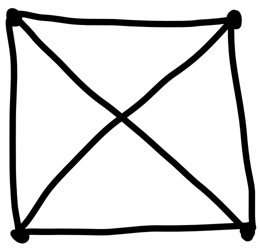
§10.7: Planar graphs

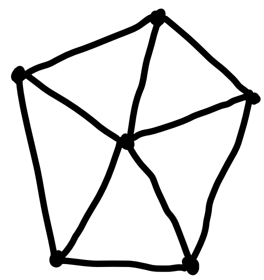
Def: A graph is called <u>planar</u> if it can be drawn w/out edges crossing

Such a drawing is called a <u>planar representation</u>

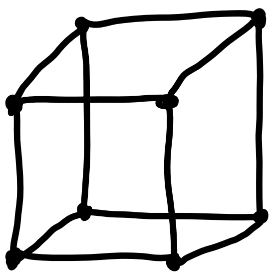
Class activity: are the following graphs planar?

a) 

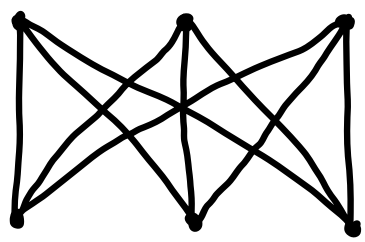
K_4

b) 

W_5

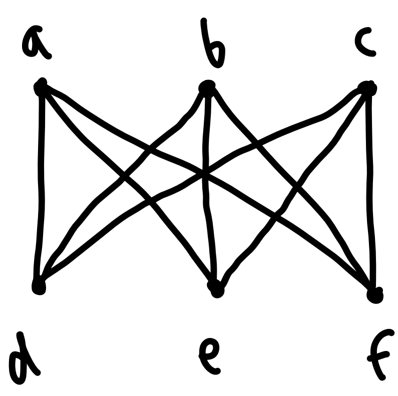
c) 

Q_3

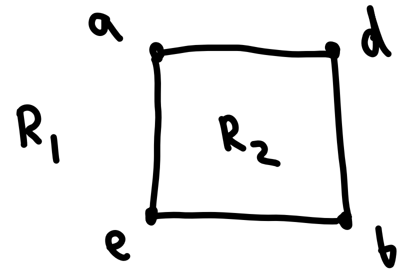
d) 

$K_{3,3}$

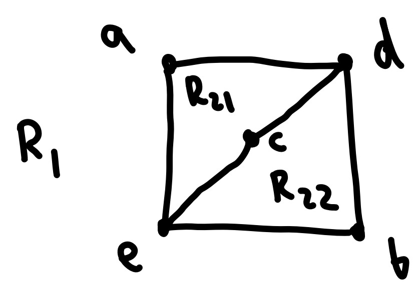
Ex 3: $K_{3,3}$ is nonplanar



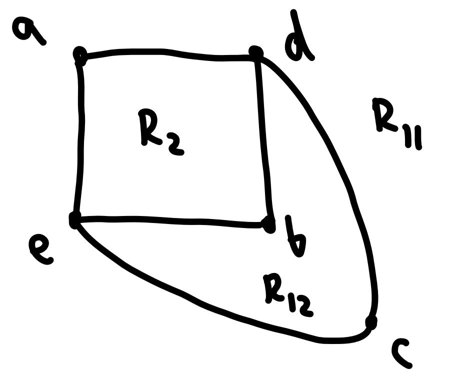
Pf: Suppose $K_{3,3}$ has a planar representation. Then the induced subgraph formed by a, b, d, e is C_4:



Two possible ways to add in c:



or



Finally, we need to place f and connect it to a, b, and c. To do so w/out edge crossings, f must lie in a region w/ a, b, c on the boundary.

For the left diagram:

Region R_1: a, b on bdy, c not

Region R_{21}: a, c on bdy, b not

Region R_{22}: b, c on bdy, a not

For the right diagram:

Region R_2: a, b on bdy, c not

Region R_{11}: a, c on bdy, b not
Region R_{12}: b, c on bdy, a not

Therefore, no such region exists, so $K_{3,3}$ is not planar. $\square$

HW: similar argument for K_5

Now count vertices (v), edges (e), and regions (r) in the planar representations above

planar rep'n of a

Euler's formula: For every connected planar graph,

$$v - e + r = 2$$

Ex 4: Suppose that a conn. planar graph has 20 vertices, each of deg. 3. How many regions does a planar representation have?

Sol'n:

$$v = 20$$

$$e = \frac{1}{2} \sum_{v \in V} \deg(v) = \frac{1}{2} \cdot 20 \cdot 3 = 30$$

Apply Euler's formula:

$$20 - 30 + r = 2 \rightsquigarrow r = 12$$

Notice that if the number of vertices is fixed, more edges means more regions

Cor 1: If G is a $\underbrace{\text{connected planar simple}}_{\text{notice all the adjectives!}}$ graph and $v \geq 3$, then $e \leq 3v-6$

Pf: Let the degree be the number of edges on the bdy of the region (counted twice if both "sides" of the edge are in the same region)

Since the graph is simple, if $e \geq 2$, each region has degree ≥ 3, and since each edge contributes a total of 2 to the degree of regions,

$$2e = \sum_{\substack{\text{regions}\\ R}} \deg(R) \geq 3r, \quad \text{so } r \leq \tfrac{2}{3}e$$

By Euler's formula,

$$2 = v - e + r \leq v - e + \tfrac{2}{3}e = v - \tfrac{1}{3}e,$$

so $e \leq 3v-6$. $\square$

Ex 5: Let $G = K_5$.

$v = 5$

$e = \binom{5}{2} = \frac{5!}{2!\,3!} = 10$

$3v - 6 = 9 < 10$, so K_5 is not planar

Cor 2: If G is a conn. simple planar graph, then G has a vertex of deg ≤ 5.

Pf: If $v=1$ or 2, this is clearly true. Suppose that $v \geq 3$ and $\deg(u) \geq 6$ for all $u \in V$. Then, by the Handshake Theorem:

$$2e = \sum_{u \in V} \deg(u) \geq \sum_{u \in V} 6 = 6v$$

so $e \geq 3v > 3v-6$, which is Impossible. $\square$