Announcements

Midterm 2: Wednesday in class (50 minutes)

- Covers through Chapter 9.3
- Reference sheet allowed (one A4 sheet, both sides)
- Practice problem sol'ns posted
- See policy email

Midterm 2 Review

(Partial) list of topics:

Propositional logic, sets: truth tables, operations, identities, etc.

Functions: domain etc., inj/surj/bij, composition, inverses

Algorithms: properties, write/perform, searching/sorting/greedy change

Big-O: pfs & heuristics

Induction: mathematical vs. strong, various examples

Counting

- Sum/product/subtraction/division rules
- (Generalized) pigeonhole principle
- Permutations/combinations, and generalized versions
- Binomial coeffs., identities, and the binom. thm

Probability

- Def'ns (event, sample space, etc.)
- Basic examples (e.g. coins, dice, cards)

Independence

Bernoulli trials

Conditional probability & Bayes' Thm.

Recurrence rel'ns

Basic ideas, examples

Linear (in)homogeneous rec. rel'ns, and how to solve e.g. $\boxed{\text{Thm. 6}}$

Inclusion-Exclusion & applications (integer eq'ns, derangements)

Relations

Properties: reflexive, symmetric, antisymmetric, transitive

Operations: complement, inverse, composition

Representing relations: matrix, digraph

Other tips:

Look at HW, quizzes, lecture notes, textbook, other problems

Pfs (for all topics, but particularly where we've done pfs)

Methods (e.g. sticks and stones)

Practice!

Examples:

1) 6.4.29: Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k} = n\,2^{n-1}$$

Pf: We consider the task of choosing a committee (of any size) out of n people, and choosing one committee member to be the chair.

Method 1:

- Choose the size k of the committee
- Choose the k members of the committee — $\binom{n}{k}$ ways
- Choose one of these members to be the chair — k ways

Total num. ways: $\sum_{k=0}^{n} \binom{n}{k} k = \sum_{k=1}^{n} \binom{n}{k} k$

Method 2:

- Choose the chair — n ways
- For each of the $n-1$ remaining people, choose whether or not they're on the committee — 2^{n-1} ways

Total num. ways: $n 2^{n-1}$

Therefore, since these methods count the same set, the num. ways must be the same in each case, i.e.

$$\sum_{k=1}^{n} \binom{n}{k} k = n 2^{n-1} \qquad \square$$

Notice that an algebraic approach also works. We'll do the case n: even for simplicity:

$$\sum_{k=1}^{n} k \binom{n}{k} = \sum_{k=1}^{n} \binom{n}{k} + \sum_{k=2}^{n} \binom{n}{k} + \cdots + \sum_{k=n-1}^{n} \binom{n}{k} + \sum_{k=n}^{n} \binom{n}{k}$$

$$= \sum_{k=1}^{n}\binom{n}{k} + \sum_{k=2}^{n}\binom{n}{k} + \cdots + \sum_{k=n-1}^{n}\binom{n}{n-k} + \sum_{k=n}^{n}\binom{n}{n-k}$$

$$= \frac{n}{2}\sum_{k=0}^{n}\binom{n}{k} = \frac{n}{2}\cdot 2^n = n2^{n-1}$$

2) 7.2.27: Consider a family w/ n children (each gender chosen by coin flip). Let

$E = \{\geq 1\,G \text{ and } \geq 1\,B\}$

$F = \{\leq 1\,B\}$

Are E and F independent if

a) $n=2$ b) $n=4$ c) $n=5$?

Soln: Recall that E & F are indep. if $p(E\cap F) = p(E)\,p(F)$

a) We can do this explicitly

GG : E false, F true BG : E true, F true

GB : E true, F true BB : E false, F false

$p(E) = \frac{1}{2}$ $p(F) = \frac{3}{4}$ $p(E)p(F) = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$

$p(E\cap F) = \frac{2}{4} \neq \frac{3}{8} = p(E)p(F)$

So E, F are not indep.

b) Here the sample space S has size $|S| = 2^4$ by the prod. rule.

E is false in exactly the following cases: GGGG, BBBB

so $p(E) = 1 - p(\overline{E}) = 1 - \frac{2}{16} = \frac{7}{8}$

F is true in exactly the following cases: GGGG, GGGB, GGBG, GBGG, BGGG

so $p(F) = \frac{5}{16}$

$E \wedge F$ is true in exactly the following cases: GGGB, GGBG, GBGG, BGGG

so $p(E \wedge F) = \frac{4}{16}$

$p(E)p(F) = \frac{7}{8} \cdot \frac{5}{16} = \frac{35}{128} \neq \frac{4}{16} = p(E \wedge F)$

So E, F are not indep.

c) Here the sample space S has size $|S| = 2^5$ by the prod. rule.

E is false in exactly the following cases: GGGGG, BBBBB

so $p(E) = 1 - p(\overline{E}) = 1 - \frac{2}{32} = \frac{15}{16}$

F is true in exactly the following cases: GGGGG, GGGGB, GGGBG, GGBGG, GBGGG, BGGGG

so $p(F) = \frac{6}{32}$

$E \wedge F$ is true in exactly the following cases: GGGGB, GGGBG, GGBGG, GBGGG, BGGGG

so $p(E \wedge F) = \frac{5}{32}$

$p(E)p(F) = \frac{15}{16} \cdot \frac{6}{32} = \frac{45}{256} \neq \frac{5}{32} = p(E \cap F)$

So E, F are not indep.

3) 8.2.11: Solve the linear homog. rec. rel'n

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \quad L_1 = 1$$

Sol'n: Characteristic eqn: $r^2 - r - 1 = 0$

By the quadratic formula, $r = \frac{1 \pm \sqrt{5}}{2}$, so by Thm. 1 of §8.2,

$$L_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for some}$$

constants α_1 and α_2. Plugging in the initial conds.:

$$2 = L_0 = \alpha_1 + \alpha_2$$

$$1 = L_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = \frac{\alpha_1 + \alpha_2}{2} + \frac{\sqrt{5}(\alpha_1 - \alpha_2)}{2}$$

Solving these eqns, we obtain $\alpha_1 = \alpha_2 = 1$, so

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

4) 6.5.30: How many different strings can be made from the letters in Mississippi, using all the letters?

Sol'n: 11 letters:

1 M

4 I's

4 S's

2 P's

Arrangements: $\binom{11}{1,4,4,2} = \frac{11!}{1!\,4!\,4!\,2!} = 34650$

5) 9.1.44: How many relations on $A = \{0,1\}$ are symmetric?

Sol'n: Rel'ns on A are subsets of $A \times A = \{(0,0), (0,1), (1,0), (1,1)\}$

16 subsets of $A \times A$

If $R \subseteq A \times A$ is symmetric, it either includes both $(0,1)$ and $(1,0)$, or neither.

To choose a symmetric relation R, choose whether R contains:

$(0,0)$ $\quad$ $(1,1)$ $\quad$ $(0,1)$ & $(1,0)$

2 options · 2 options · 2 options = 8 symmetric rel'ns on A