

Announcements

Quiz 4 today!

Midterm 2: Wednesday in class

Covers through roughly §9.3

Reference sheet allowed (one A₄ sheet w/ writing on both sides)

See policy email (practice problems, etc.)

Recall: A relation from A to B is a subset R of $A \times B$.

Write $a R b$ to mean $(a, b) \in R$

R is a function if every elt. of A appears exactly once in R .

Often, we care about relations from A to A ("on A ")

Class activity: Let $A = \{1, 2, 3\}$

Match the symbols w/ the relations

i) \leq ii) $>$ iii) $=$ iv) $|$ ("divides")

a) $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$

b) $\{(1, 1), (2, 2), (3, 3)\}$

c) $\{(2, 1), (3, 1), (3, 2)\}$

d) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

Ex 6: If $|A|=n$, there are 2^{n^2} relations on A , n^n of which are functions.

e.g. $n=3$: 512 relations, 27 functions

$n=4$: 65536 relations, 256 functions

Properties:

- R is reflexive if aRa for all $a \in A$
- R is symmetric if whenever aRb , then bRa
- R is antisymmetric if whenever aRb and $a \neq b$, then $b \not R a$
- R is transitive if whenever aRb and bRc , then aRc

Class activity: For each of these relations on \mathbb{Z} , determine whether it has each of the above properties:

$$R_1: \leq$$

$$R_2: >$$

$$R_3: =$$

$$R_4: |$$

$$R_5: \{(a,b) \mid a=b \text{ or } a=-b\}$$

$$R_6: \{(a,b) \mid a=b+1\}$$

$$R_7: \{(a,b) \mid a+b \leq 3\}$$

$$R_8: \{(1,2), (2,3), (3,1)\}$$

$$R_9: \{(1,2), (2,3), (1,3)\}$$

$$R_{10}: \{(1,1), (2,2), (3,3)\}$$

$$R_{11}: \{(1,5), (5,1), (2,2), (2,3), (3,2)\}$$

$$R_{12} = R_3 \cup R_8$$

union/intersection

$$R_{13} = R_3 \cap R_8$$

as sets

Operations:

- Complement: $\bar{R} = \{(a,b) \in A \times B \mid (a,b) \notin R\}$
- Inverse: $R^{-1} = \{(b,a) \mid (a,b) \in R\}$ relation from B to A
- Composition: If $R \subseteq A \times B$, $S \subseteq B \times C$, then

$S \circ R = \{(a,c) \in A \times C \mid \text{there exists } b \in B \text{ s.t. } (a,b) \in R, (b,c) \in S\}$
relation from A to C

Ex: $A = \{1, 2, 3\}$, $B = \{(1, 2, 3, 4)\}$

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4)\}$$

Class activity: find

$$\bar{R}, R^{-1}, R \circ R^{-1}, \text{ and } R^{-1} \circ R$$

§9.3: Representing relations

Option 1: Use a matrix!

$$A = \{a_1, a_2, \dots, a_m\}$$

$$B = \{b_1, \dots, b_n\}$$

$R \subseteq A \times B$
relation

Then R can be represented by the matrix

$$M_R = [m_{ij}] \text{ where } m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{otherwise} \end{cases}$$

Ex 1: $A = \{1, 2, 3\}$ $B = \{1, 2\}$

$$R = \{(2, 1), (3, 1), (3, 2)\}$$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Class activity (Ex 3):

Let R be a relation on A s.t.

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Find R . Is it reflexive?

Symmetric? Antisymmetric?

If R & S are relns on A , then

$$M_{R \cup S} = M_R \cup M_S \leftarrow \begin{cases} \text{an entry is } 1 \text{ if the} \\ \text{corresp. entry in either} \\ M_R \text{ or } M_S \text{ is } 1 \end{cases}$$

$$M_{R \cap S} = M_R \cap M_S \leftarrow \begin{cases} \text{same, but replace "either"} \\ \text{with "both"} \end{cases}$$

$$M_{S \circ R} = M_R \odot M_S \leftarrow \begin{cases} \text{boolean prod: take matrix} \\ \text{product, then change any pos.} \\ \text{num. to } 1 \end{cases}$$

note: order reversed!

Ex: If $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$M_{R \cup S} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad M_{R \cap S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_R M_S = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix},$$

$$\text{so } M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Option 2: Use a graph!

← not the x,y
kind of graph

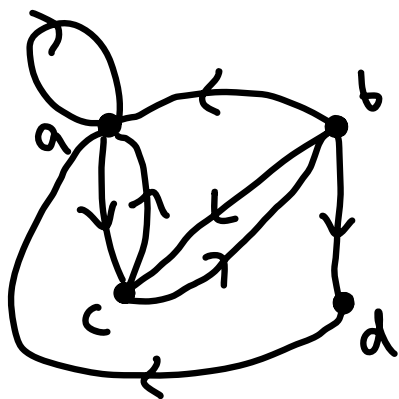
Def: A directed graph or digraph consists of a set V of vertices and a set E of edges. Each edge goes from one vertex to another

Let R be a relation on A . To make a digraph for R ,
 V : vertices labelled by elts. of A

E : edge going from a to b if $(a,b) \in R$

Ex 8: $A = \{a, b, c, d\}$

$R = \{(a,a), (a,c), (b,a), (b,c), (b,d), (c,a), (c,b), (d,a)\}$



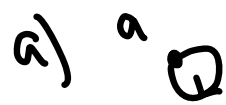
Reflexive: loop at every vertex

Symmetric: edges always come in opposite-direction pairs

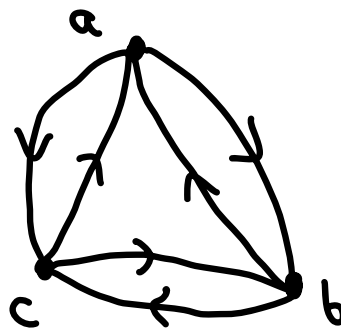
Antisymmetric: edges never come in opposite-direction pairs

Transitive: For every length-2 path, there is a single edge w/ the same start and end

Class activity: Are the following rel'n's reflexive/symmetric/antisymmetric/transitive?
 Draw the corresponding matrix/digraph.



b)



c)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$