# Announcements

Quiz 4: this Wed, in class

Midterm 2: Wed 4/1, in class (policy email coming soon)

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Recall: inclusion - exclusion

$$|A_1 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$
$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots A_n|$$

3 sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

Last time: used this to count sol'ns of

$$x_1 + x_2 + x_3 = 11$$

where $x_1, x_2, x_3 \in \mathbb{N}$ and $x_1 \le 3$, $x_2 \le 4$, $x_3 \le 6$

Ex 2: How many surjective functions are there from $A \to B$ if $|A| = 6$, $|B| = 3$? What about if $|A| = m$, $|B| = n$?

Sol'n: Let $B = \{x, y, z\}$

$U = \{\text{functions } f: A \to B\}$

$P = \{f \in U \mid x \notin f(A)\}$ range of f

$Q = \{f \in U \mid y \notin f(A)\}$

$R = \{f \in U \mid z \notin f(A)\}$

Want: $|U \setminus (P \cup Q \cup R)|$

$|U| = 3^6 \qquad |P| = |Q| = |R| = 2^6$

$|P \cap Q| = |P \cap R| = |Q \cap R| = 1^6 \qquad |P \cap Q \cap R| = 0$

So $|U \setminus (P \cup Q \cup R)| = |U| - |P \cup Q \cup R|$

$= |U| - |P| - |Q| - |R| + |P \cap Q| + |P \cap R| + |Q \cap R| - |P \cap Q \cap R|$

$= 3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 - 0$

$= 729 - 192 + 3 = 540$

Similarly, if $|A|=m$, $|B|=n$, there are

$$\binom{n}{0}n^m - \binom{n}{1}(n-1)^m + \underbrace{\binom{n}{2}}_{\substack{\text{choose the}\\ \text{2 elts. from}\\ \text{B to exclude}}}(n-2)^m - \binom{n}{3}(n-3)^m + \dots + (-1)^{n+1}\binom{n}{n-1}1^m$$

Def: A <u>derangement</u> is a permutation of objects that leaves no object in its initial position

e.g. 21453 ✓ 3142$\underline{5}$ ✗ 54$\underline{3}$21 ✗ $\underline{1}\underline{2}\underline{3}\underline{4}\underline{5}$ ✗

Ex 4: What is the number of derangements of n objects?

Soln: $U = \{\text{all permutations of } n \text{ objects}\}$

$A_i = \{\text{permutations where } i \text{ is in the } i\text{th spot}\}$

Want: $|\overline{A_1 \cup \dots \cup A_n}|$

$|U| = n!$

$|A_i| = (n-1)!$ since the ith spot is fixed

$|A_i \cap A_j| = (n-2)!$ since the ith, jth spots are fixed

$\vdots$

$$|\overline{A_1 \cup \cdots \cup A_n}| = |U| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|$$

$$= n! - \sum_i (n-1)! + \sum_{i<j} (n-2)! + \cdots + (-1)^n 0!$$

$$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \cdots + (-1)^n \binom{n}{n} 0!$$

$$= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! + \cdots + (-1)^n \frac{n!}{0!n!} 0!$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Follow-up: what is the probability that a randomly chosen permatation of a set of size $n$ is a derangement?

$E = \{\text{derangements}\}$  $S = \{\text{all permatations}\}$

$$p(E) = \frac{|E|}{|S|} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

Something to think about if you've seen Taylor series: what happens to this probability if $n$ gets very large?

## §9.1: Relations

Def: Let $A$ and $B$ be sets. A <u>relation</u> from $A$ to $B$ is a subset $R$ of $A \times B$.

Write $aRb$ to mean $(a,b) \in R$

Ex: $A = \{a, b, c, d, e\}$ students

$B = \{C1, C2, DM\}$ math classes

$$R = \{(a, C1), (a, DM), (b, DM), (c, C2), (c, DM), (e, C1), (e, C2), (e, DM)\}$$

a is taking C1 & DM

b is taking DM

c is taking C2 & DM

d is taking nothing

e is taking $C_1$, $C_2$, & DM

$$a R C_1, \; a R C_2, \text{ etc.}$$

$R$ is a function if every elt. of $A$ appears exactly once in $R$.

Often, we care about relations from $A$ to $A$ ("on $A$")

Class activity: Let $A = \{1,2,3\}$

Match the symbols w/ the relations

i) $\leq$ ii) $>$ iii) $=$ iv) $|$ ("divides")

a) $\{(1,1), (1,2), (1,3), (2,2), (3,3)\}$

b) $\{(1,1), (2,2), (3,3)\}$

c) $\{(2,1), (3,1), (3,2)\}$

d) $\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$