Announcements

HW 7 posted (due Wed. 3/25)

---

§8.2: Solving Linear recurrence relations

Recall: A <u>sequence</u> is an infinite list of numbers

$$a_1, a_2, a_3, \ldots$$

doesn't need to start w/ a_1

A <u>recurrence relation</u> is a formula for a_n in terms of (some of) $a_1, a_2, \ldots, a_{n-1}$.

Given a recurrence rel'n and some <u>initial condition(s)</u> (value of at least a_1) we try to <u>solve</u> the recurrence rel'n by giving an explicit formula (not a recurrence rel'n) for a_n.

Today we'll learn how to solve one particular kind of recurrence rel'n

Def: A linear homogeneous recurrence rel'n of degree k w/ constant coefficients is a recurrence rel'n of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, \ldots, c_k \in \mathbb{R}$, $c_k \neq 0$.

Ex: $f_n = f_{n-1} + f_{n-2}$ ✓ linear homogeneous of deg. 2

$H_n = 2H_{n-1} + 1$ x not linear homog.

$C_{n+1} = C_0C_n + C_1C_{n-1} + \cdots + C_nC_0$ x not linear homog.

Ex: $a_n = c\,a_{n-1}$, $a_1 = c$ linear homog. of deg 1

Then, $a_2 = a_1 = c^2$

$a_3 = a_2 = c^3$

$\vdots$

$a_n = c^n$

Let's look for sol'ns of the form

$a_n = r^n$ for <u>any</u> linear homog. recurrence rel'n

Need to find r

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad a_n = r^n$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

Divide by r^{n-k}:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

This is called the <u>characteristic equation</u> for $\{a_n\}$

We can only have $a_n = r^n$ if r is a root of the characteristic eqn.

In fact,

Thm: Suppose that

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$

Then $\{a_n\}$ is a sol'n of the recurrence rel'n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for some values $\alpha_1, \alpha_2, \ldots, \alpha_k$

Ex 4: $f_n = f_{n-1} + f_{n-2}$

Degree $k=2$, $c_1 = c_2 = 1$

So the characteristic eqn. is:

$$r^2 - r - 1 = 0$$

Roots: $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

So we must have

$$f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for some } \alpha_1, \alpha_2 \in \mathbb{R}.$$

To find α_1, α_2, plug in initial conditions $f_1 = f_2 = 1$
(so $f_0 = 0$)

$f_0 = 0$, so $\quad 0 = \alpha_1 + \alpha_2$

$f_1 = 1$, so $\quad 1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)$

Sol'n: $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$

Therefore,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Ex 3: Solve the recurrence rel'n

$a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 2$, $a_1 = 7$

Sol'n: Characteristic eqn: $r^2 - r - 2 = 0$
Roots: $r_1 = 2$, $r_2 = -1$

So $a_n = \alpha_1 2^n + \alpha_2(-1)^n$

Plug in to find α_1, α_2:

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1(2) + \alpha_2(-1)$

$\alpha_1 = 3, \ \alpha_2 = -1$

So

$a_n = 3 \cdot 2^n - (-1)^n$

Thm (degree 2 case w/ repeated roots):

Thm: Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has a repeated root r.

Then $\{a_n\}$ is a sol'n of the recurrence rel'n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = (\alpha + \beta n) r^n$$

for some values α, β

Ex: $a_n = 4a_{n-1} - 4a_{n-2}$, $a_0 = a_1 = 3$

Char. eqn: $r^2 - 4r + 4 = 0$

(Double) sol'n: $r = 2$

So $a_n = (\alpha + \beta n) 2^n$

$a_0 = 3 = \alpha$

$a_1 = 3 = (\alpha + \beta) 2$

$\alpha = 3$, $\beta = -\frac{3}{2}$

$a_n = (3 - \frac{3}{2} n) 2^n$

One last modification: let's turn this <u>inhomogeneous</u> by adding a function of n:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k} + F(n)$$

Thm: Suppose you can find <u>any</u> particular sol'n $\{a_n^{(p)}\}$ to this eqn.

Then every sol'n is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is a sol'n to the homog. eqn.:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k}$$

Ex 12: Find all soln's to the recurrence reln

$$a_n = 6a_{n-1} - 9a_{n-2} + n\,3^n$$

Soln:

Homog. eqn:

$$a_n = 6a_{n-1} - 9a_{n-2}$$

Char. eqn:

$$r^2 - 6r - 9 = 0$$

$r = 3$ (double root!)

So the soln's to the homog. eqn are:

$$a_n^{(h)} = (\alpha + \beta n)3^n$$

Now we need a particular soln.

Theorem 6 in Rosen $\rightarrow$ $a_n^{(p)}$ has the form

$$a_n^{(p)} = n^2(p_1 n + p_0)3^n$$

Solve for p_0 and p_1 by plugging into the recurrence reln:

$$a_n = 6a_{n-1} - 9a_{n-2} + n\,3^n$$

$$n^2(p_1 n + p_0)3^n = 6(n-1)^2(p_1(n-1)+p_0)3^{n-1} - 9(n-2)^2(p_1(n-2)+p_0)3^{n-2} + n\,3^n$$

Match $n3^n$ coeffs: $0 = 1 - 6p_1$

3^n coeffs: $0 = 6p_1 - 2p_0$

$p_1 = \frac{1}{6}$, $p_0 = \frac{1}{2}$

So $a_n^{(p)} = n^2\left(\frac{1}{6}n + \frac{1}{2}\right)3^n$

General soln: $a_n = a_n^{(h)} + a_n^{(p)} = (\alpha + \beta n)3^n + n^2\left(\frac{1}{6}n + \frac{1}{2}\right)3^n$